QUIZ #26, 11/27/07

MATH 54, FALL 2007

Show your work and justify your answers! Feel free to use both sides.

Name:

1. (5pts) (a) Find $\exp\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right)$.

We have $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = I + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, so

$$\begin{aligned} \exp\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} t\right) &= \exp(It)\exp\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} t\right) \\ &= e^t I \left(I + t\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + [\textit{rest are zero}]\right) \\ &= e^t \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \end{aligned}$$

Thus, plugging in $t = 1$, we have $\exp\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} e & 0 \\ e & e \end{bmatrix}$.

(b) Find the general solution to $\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\mathbf{x}$.

The general solution is a linear combination of the columns of $\exp\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} t\right) = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$.

2. (5pts) (a) Find $\exp\left(\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}\right)$.

The eigenvalues are both -2.
We have $\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} = -2I + \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$, so

$$\begin{aligned} \exp\left(\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} t\right) &= \exp(-2It)\exp\left(\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} t\right) \\ &= e^{-2t} I \left(I + t\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} + [\textit{rest are zero}]\right) \\ &= e^{-2t} \begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix} = \begin{bmatrix} (1+2t)e^{-2t} & te^{-2t} \\ -4te^{-2t} & (1-2t)e^{-2t} \end{bmatrix} \end{aligned}$$

Thus, plugging in $t = 1$, we have $\exp\left(\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}\right) = \begin{bmatrix} 3e^{-2} & e^{-2} \\ -4e^{-2} & -e^{-2} \end{bmatrix}$.

(b) Find the general solution to $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}\mathbf{x}$.

The general solution is a linear combination of the columns of

$$\exp\left(\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} t\right) = \begin{bmatrix} (1+2t)e^{-2t} & te^{-2t} \\ -4te^{-2t} & (1-2t)e^{-2t} \end{bmatrix}.$$