QUIZ #25, 11/20/07

MATH 54, FALL 2007

Show your work and justify your answers! Feel free to use both sides.

Name:

1. (5pts) (a) Find the general solution to the system $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

We first diagonalize the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Its characteristic polynomial is $\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$, so its eigenvalues are 1 and 3.
We find the corresponding eigenvectors:
$A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. This has kernel spanned by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
$A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. This has kernel spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
Thus the general solution is

$$C_1 e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -C_1 e^t + C_2 e^{3t} \\ C_1 e^t + C_2 e^{3t} \end{bmatrix}.$$

(b) Find a solution with $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

This is easy because it's true for one of our eigenvectors, but let's go through the motions:
Plug in 0 for t in the general solution and we get $\begin{bmatrix} -C_1 + C_2 \\ C_1 + C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
That is, $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We can solve this to get $C_1 = 0$ and $C_2 = 1$.
Thus the solution is $\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$.

2. (5pts) Find the general solution to the system $\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$.

Again we first diagonalize the matrix $A =$. To find the characteristic polynomial, we expand along the first row (or column) in $A - \lambda I$ and get $(-\lambda - 2)(\lambda^2 - 2\lambda + 2)$. Thus the eigenvalues are -2 and $1 \pm i$.
Due to the block form of the matrix (or just by writing down $A + 2I$, we see that the eigenvectors with eigenvalue -2 are the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Now we find the eigenvectors with eigenvalue $1+i$. We have $A-(1+i)I = \begin{bmatrix} -3-i & 0 & 0 \\ 0 & 1-i & -1 \\ 0 & 2 & -1-i \end{bmatrix}$.

If we subtract $1+i$ times the second row from the third, this becomes $\begin{bmatrix} -3-i & 0 & 0 \\ 0 & 1-i & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

Thus the kernel is the span of

$$\begin{bmatrix} 0 \\ 1 \\ 1-i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \vec{v} + i\vec{w}$$

Thus the general solution is

$$C_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 e^t \left(\cos t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right) + C_3 e^t \left(\cos t \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right).$$