QUIZ SOLUTIONS #24, 11/15/07

MATH 54, FALL 2007

1. (6 pts) Suppose $\mathbf{x}_1 = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 4e^{5t} \\ e^{5t} \end{bmatrix}$ are solutions to a differential equation of the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(a) Compute the Wronskian of the solution set $\{\mathbf{x}_1, \mathbf{x}_2\}$.

$$W(t) = \det \begin{bmatrix} e^{2t} & 4e^5 t \\ -e^{2t} & e^{5t} \end{bmatrix} = 5e^{7t}.$$

(b) Is $\{\mathbf{x}_1, \mathbf{x}_2\}$ necessarily a fundamental solution set? If so, what is the general solution?

The Wronskian is not zero, so yes it is a fundamental solution set. (In other words, we have two linearly independent solutions, which is as many as we expect, given that we're dealing with vectors with two entries.)The general solution is $c_1 \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 4e^{5t} \\ e^{5t} \end{bmatrix}$.

(c) Find a solution with $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Plugging in $t = 0$ to the general solution, we get $c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Solving this gives $c_1 = -\frac{1}{5}$ and $c_2 = \frac{4}{5}$. Thus

$$\mathbf{x} = -\frac{1}{5} \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} + \frac{4}{5} \begin{bmatrix} 4e^{5t} \\ e^{5t} \end{bmatrix}.$$

2. (4 pts) Consider the differential equation $y''(t) + by'(t) + cy(t) = g(t)$.

(a) Write this in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}.$$

(b) When is zero an eigenvalue of $\mathbf{A}$? (State your answer in terms of b, c, and $g(t)$.)

Zero is an eigenvalue of A when $\det(A - 0I) = 0$. That is, when $\det(A) = 0$. We see that $\det(A) = c$, so we simply need $c = 0$.