QUIZ SOLUTIONS #20, 11/1/07

MATH 54, FALL 2007

1. (6 pts) Is $A = \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}$ diagonalizable? If so, find a change of basis matrix S and a diagonal matrix D such that $S^{-1}AS = D$. If not, explain why not.

Its characteristic polynomial is $\lambda^2 - 12\lambda + 35$. The roots of this are $\lambda_1 = 5$ and $\lambda_2 = 7$. Thus it is diagonalizable (because the roots are distinct) and is similar to $D = \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}$. (The 5 and 7 can be swapped.)
To find the change-of-basis matrix, we need to find the eigenvectors.

$$\mathrm{Eig}_5 = \ker(A - 5I) = \ker \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

$$\mathrm{Eig}_7 = \ker(A - 7I) = \ker \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

Thus the change-of-basis matrix is $S = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ (that is, we put the eigenvectors as the columns).

2. (4 pts) For what values of a, b, c, and d is the matrix $\begin{bmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{bmatrix}$ diagonalizable?

This is in block form with a 2×2 block (the upper left) and a 1×1 block (the d), so it won't matter what d is and the only question is whether we can diagonalize the 2×2 block $B = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ (because then the whole matrix will be diagonal).
The eigenvalues of B are a and c, so if $a \neq c$, then B is diagonalizable because it has distinct eigenvalues. If $a = c$, then we'll need b to be zero, otherwise there will only be one eigenvector for B with eigenvalue $a = c$ (check this!).
Thus the answer is: the whole matrix is diagonalizable if $a \neq c$ or if $b = 0$.