

QUIZ SOLUTIONS #19, 10/30/07

MATH 54, FALL 2007

1. (7 pts) (a) Find all real eigenvalues of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ with their algebraic multiplicities.

The characteristic polynomial is $f_A(\lambda) = (1 - \lambda)(2 - \lambda)(2 - \lambda)$. Thus 1 is an eigenvalue of algebraic multiplicity one and 2 is an eigenvalue of algebraic multiplicity two.

- (b) Find all eigenvectors of A . What are the geometric multiplicities of each eigenvalue of A ?

The space of eigenvectors with eigenvalue 1 is the kernel of $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. This is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Thus the eigenvalue 1 has a geometric multiplicity of one.

The space of eigenvectors with eigenvalue 2 is the kernel of $\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. This is spanned by

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Thus the eigenvalue 2 has a geometric multiplicity of one.

- (c) Does A have an eigenbasis? If yes, find an eigenbasis for A .

No it doesn't, because the geometric multiplicities only add up to $1 + 1 = 2 < 3$. (Alternatively, there's an eigenvalue [namely 2] whose geometric multiplicity doesn't equal its algebraic multiplicity.)

2. (3 pts) Suppose A is an $n \times n$ matrix that has an eigenbasis and all of its eigenvalues have absolute value less than one. Consider some vector \vec{v} (not necessarily an eigenvector). What is $\lim_{t \rightarrow \infty} A^t \vec{v}$? (Hint: Write \vec{v} as a linear combination of eigenvectors.)

Let v_1, \dots, v_n be an eigenbasis with eigenvalues $\lambda_1, \dots, \lambda_n$ (some of the λ_i 's may be the same). Write $v = a_1 v_1 + \dots + a_n v_n$. Then we have $A^t v = a_1 (\lambda_1)^t v_1 + \dots + a_n (\lambda_n)^t v_n$. Because $|\lambda_i| < 1$ (for each i), $\lim_{t \rightarrow \infty} (\lambda_i)^t = 0$. Thus $\lim_{t \rightarrow \infty} A^t v = 0$.