QUIZ SOLUTIONS #18, 10/25/07

MATH 54, FALL 2007

Show your work and justify your answers! Feel free to use both sides.

Name:

1. (4 pts) If $\overrightarrow{v}$ is an eigenvector of an $n \times n$ matrix A with eigenvalue λ, is $\overrightarrow{v}$ an eigenvector of $3A^3 + 2A - 3I_n$ (where I_n is the $n \times n$ identity matrix)? If so, what is the associated eigenvalue?

We have (doing out every step):

$$\begin{aligned}
(3A^3 + 2A - 3I_n)\overrightarrow{v} &= 3A^3\overrightarrow{v} + 2A\overrightarrow{v} - 3I_n\overrightarrow{v} = 3A^2(A\overrightarrow{v}) + 2\lambda\overrightarrow{v} - 3\overrightarrow{v} \\
&= 3A^2\lambda\overrightarrow{v} + (2\lambda - 3)\overrightarrow{v} = 3A\lambda A\overrightarrow{v} + (2\lambda - 3)\overrightarrow{v} = 3\lambda A(A\overrightarrow{v}) + (2\lambda - 3)\overrightarrow{v} \\
&= 3\lambda A(\lambda\overrightarrow{v}) + (2\lambda - 3)\overrightarrow{v} = 3\lambda^2 A\overrightarrow{v} + (2\lambda - 3)\overrightarrow{v} = 3\lambda^2\lambda\overrightarrow{v} + (2\lambda - 3)\overrightarrow{v} \\
&= (3\lambda^3 + 2\lambda - 3)\overrightarrow{v}
\end{aligned}$$

Thus $\overrightarrow{v}$ is an eigenvector of $3A^3 + 2A - 3I_n$, and its eigenvalue is $3\lambda^3 + 2\lambda - 3$.
More intuitively: A^3 has $\overrightarrow{v}$ as an eigenvector with eigenvalue λ^3, A has $\overrightarrow{v}$ as an eigenvector with eigenvalue λ, and I_n has $\overrightarrow{v}$ as an eigenvector with eigenvalue 1, and a linear combination of matrices with the same eigenvector is a matrix with that vector as an eigenvector with eigenvalue equal to the appropriate linear combination of the eigenvalues.

2. (6 pts) (a) What are the eigenvalues of the transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by rotation by 180°. (b) What are all the associated eigenvectors?

This transformation maps everything to its negative (i.e. $T(\overrightarrow{v}) = -\overrightarrow{v}$ for every vector $\overrightarrow{v}$). Thus its only eigenvalue is -1, and every vector (excluding $\overrightarrow{0}$ perhaps, which we don't count as an eigenvector) is an eigenvector with -1 as an eigenvalue.

(c) Does rotation counterclockwise by 90° have any eigenvectors?

No. No vector is sent to a multiple of itself because, for instance, every vector goes to something orthogonal to itself (and no nonzero vector is sent to zero).