QUIZ SOLUTIONS #16, 10/18/07

MATH 54, FALL 2007

1. Consider the inner product $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$ (on the linear space of continuous functions on $[-\pi,\pi]$).

(a) (3 pts) Find an orthonormal basis for $P_1 = \text{span}(1, x)$.

We have

$$u_1 = \frac{1}{\sqrt{\langle 1,1\rangle}} = \frac{1}{\sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} 1\,dx}} = \frac{1}{\sqrt{\frac{1}{\pi}2\pi}} = \frac{1}{\sqrt{2}}.$$

Next we have

$$v_2^{\perp} = x - \left\langle x, \frac{1}{\sqrt{2}}\right\rangle \frac{1}{\sqrt{2}} = x - \frac{1}{\sqrt{2}}\int_{-\pi}^{\pi}\frac{x}{\sqrt{2}}\,dx = x - 0 = x.$$

(The integral is zero either by calculating it, or by noticing that it's an integral of an odd function over a balanced interval.)

Thus

$$u_2 = \frac{x}{\sqrt{\langle x,x\rangle}} = \frac{x}{\sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx}} = \frac{x}{\sqrt{\frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}}} = \frac{x}{\sqrt{\frac{1}{\pi}\frac{2\pi^3}{3}}} = \frac{x}{\pi\sqrt{\frac{2}{3}}}.$$

(b) (3 pts) Find the projection of $\cos x$ (i.e. the function $f(x) = \cos x$) to P_1.

We calculate

$$\begin{aligned}\text{proj}_{P_1}\cos x &= \langle \cos x, u_1\rangle u_1 + \langle \cos x, u_2\rangle u_2 = \left\langle \cos x, \frac{1}{\sqrt{2}}\right\rangle \frac{1}{\sqrt{2}} + \left\langle \cos x, \frac{x}{\pi\sqrt{\frac{2}{3}}}\right\rangle \frac{x}{\pi\sqrt{\frac{2}{3}}} \\ &= \frac{1}{\sqrt{2}}\int_{-\pi}^{\pi}\frac{\cos x}{\sqrt{2}}\,dx + \frac{x}{\pi\sqrt{\frac{2}{3}}}\int_{-\pi}^{\pi}\frac{x\cos x}{\pi\sqrt{\frac{2}{3}}}\,dx\end{aligned}$$

We could do these integrals out (the first one is easy, and the second is a simple integration by parts), but we could also notice that the first integral is (a multiple of) $\cos x$ integrated over a full period, and that's zero, and the second integral is the integral of an odd function over a balanced interval, and that's zero. Thus

$$\text{proj}_{P_1}\cos x = 0.$$

Note that this just means the best linear approximation to $\cos x$ on the interval $[-\pi,\pi]$ (as determined by the norm we're working with) is just the function $f(x) = 0$. If you sketch $\cos x$ on this interval, this should look reasonable. Note that the best *quadratic* approximation to $\cos x$ on this interval *won't* be $f(x) = 0$.

2. (4 pts) Find the determinant of $\begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 3 \\ 3 & 2 & 0 \end{bmatrix}$.

We expand along the last row, as it has a zero in it. We get:

$$\det\begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 3 \\ 3 & 2 & 0 \end{bmatrix} = +3\det\begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix} + (-2)\det\begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix} + 0 = 3(-3-1) - 2(3-4) = -12 + 2 = -10.$$