QUIZ SOLUTIONS #15, 10/16/07

MATH 54, FALL 2007

Show your work and justify your answers! Feel free to use both sides.

Name:

1. (5 pts) Find the least-squares solution $\overrightarrow{x}^*$ of the system $A\overrightarrow{x} = \overrightarrow{b}$, where $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 3 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

We want to solve $A^T A \overrightarrow{x}^* = A^T \overrightarrow{b}$. Because A has no kernel (its columns are linearly independent), $A^T A$ is invertible, so we can simply use $\overrightarrow{x}^* = (A^T A)^{-1} A^T \overrightarrow{b}$. We calculate:

$$\begin{aligned} \overrightarrow{x}^* &= \left(\begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \\ &= \left(\begin{bmatrix} 2 & -2 \\ -2 & 13 \end{bmatrix} \right)^{-1} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 13 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} \\ &= \frac{1}{22} \begin{bmatrix} 50 \\ -6 \end{bmatrix} = \begin{bmatrix} 25/11 \\ 3/11 \end{bmatrix} \end{aligned}$$

2. (3 pts) Find a basis for the orthogonal complement of the kernel of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$.

Call our matrix B. We want a basis for $\ker(B)^\perp$.

We know $\mathrm{im}(A)^\perp = \ker(A^T)$ for any matrix A. Let's use $A = B^T$. Then $\mathrm{im}(B^T)^\perp = \ker(B)$. Taking the orthogonal complement of both sides (so $\ker(B)^\perp$ shows up), we get $\left(\mathrm{im}(B^T)^\perp\right)^\perp = \ker(B)^\perp$. But $(V^\perp)^\perp = V$ for any subspace V, so we get $\mathrm{im}(B^T) = \ker(B)^\perp$.

Thus we're just looking for a basis for $\mathrm{im}(B^T)$, i.e. the row-space of B. The two rows are linearly independent so we get:

$$\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right)$$

3. (2 pts) True or False? (You needn't justify your answer.) If A is an $n \times m$ matrix, then A^T is a matrix whose kernel is the span of the columns of A.

FALSE. The span of the columns of A is $\mathrm{im}(A)$, but $\ker(A^T) = \mathrm{im}(A)^\perp$, not $\mathrm{im}(A)$.