QUIZ SOLUTIONS #13, 10/9/07

MATH 54, FALL 2007

1. (5 pts) Perform the Gram-Schmidt process on $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Call the vectors v_1, v_2, and v_3. We use the Professor's method (i.e. we normalize at the end). That means we leave v_1 alone (to be normalized at the end) and work on v_2:

$$v_2^{\perp} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{||v_1||^2}\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - \frac{4}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4/9 \\ -8/9 \\ 10/9 \end{bmatrix}.$$

Now we leave normalizing this to the end and work on v_3:

$$\begin{aligned} v_3^{\perp} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{||v_1||^2}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{1}{||v_2^{\perp}||^2}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -4/9 \\ -8/9 \\ 10/9 \end{bmatrix}\right)\begin{bmatrix} -4/9 \\ -8/9 \\ 10/9 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{-2/9}{\frac{1}{9^2}(16+64+100)}\begin{bmatrix} -4/9 \\ -8/9 \\ 10/9 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -5/9 \\ -10/9 \\ -10/9 \end{bmatrix} + \begin{bmatrix} -4/90 \\ -8/90 \\ 10/90 \end{bmatrix} \\ &= \begin{bmatrix} 36/90 \\ -18/90 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/5 \\ -1/5 \\ 0 \end{bmatrix} \end{aligned}$$

Now we normalize:

$u_1 = \frac{v_1}{||v_1||} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$; $u_2 = \frac{v_2^{\perp}}{||v_2^{\perp}||} = \frac{1}{\frac{1}{9}\sqrt{16+64+100}}\begin{bmatrix} -4/9 \\ -8/9 \\ 10/9 \end{bmatrix} = \begin{bmatrix} -4/\sqrt{180} \\ -8/\sqrt{180} \\ 10/\sqrt{180} \end{bmatrix}$;

$u_3 = \frac{v_3^{\perp}}{||v_3^{\perp}||} = \frac{1}{\frac{1}{5}\sqrt{4+1}}\begin{bmatrix} 2/5 \\ -1/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}$

2. (5 pts) Find an orthonormal basis for the plane given by $x_1 + 3x_2 + 2x_3 = 0$.

A basis for the plane is, for example, $v_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ (find this by finding the kernel of $[-3\ 1\ 0]$). You'll get a different answer if you choose a different basis. Additionally, you'll get a

different answer if you choose this basis, but swap v_1 and v_2! (Both of these are fine; this is just to tell you that the answer here might not match what you got.)
We perform Gram-Schmidt on this to find an orthonormal basis for the plane. For variety, we'll use the book's method (of normalizing as we go along).

$$u_1 = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{9+1}} \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \\ 0 \end{bmatrix}$$

$$\begin{aligned} v_2^\perp &= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \\ 0 \end{bmatrix} \right) \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{6}{\sqrt{10}} \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -2/10 \\ -6/10 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/5 \\ -3/5 \\ 1 \end{bmatrix} \end{aligned}$$

Thus $u_2 = \frac{v_2^\perp}{||v_2^\perp||} = \frac{1}{\frac{1}{5}\sqrt{1+9+25}} \begin{bmatrix} -1/5 \\ -3/5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{35} \\ -3/\sqrt{35} \\ 5/\sqrt{35} \end{bmatrix}$