QUIZ SOLUTIONS #11, 10/2/07

MATH 54, FALL 2007

Show your work and justify your answers! Feel free to use both sides.

Name:

1. Let $T(f) = 2f' + f''$ from P_2 to P_2.

(a) (3 pts) Find the matrix of T with respect to the usual basis $(1, t, t^2)$.

We have $T(1) = 0 = 0\cdot 1 + 0\cdot t + 0\cdot t^2$, $T(t) = 2 = 2\cdot 1 + 0\cdot t + 0\cdot t^2$, and $T(t^2) = 4t + 2 = 2\cdot 1 + 4\cdot t + 0\cdot t^2$. Thus the matrix for T in the basis $(1, t, t^2)$ is $\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

(b) (3 pts) Find the kernel **and** the image of T.

A basis for the image is $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ in terms of the basis $(1, t, t^2)$. More abstractly, it's 2, $2 + 4t$. Alternatively, another basis is simply 1, t.

If we put the matrix in reduced row-echelon form, we get $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, letting x_1, x_2, and x_3 be coordinates with respect to the basis $(1, t, t^2)$, we have $x_2 = x_3 = 0$, and $x_1 = s$ is a free parameter. Thus the kernel consists of vectors of the form $s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so a basis for the kernel is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in terms of the basis $(1, t, t^2)$. More abstractly, a basis for the kernel is just the polynomial 1 (i.e. $f(t) = 1$).

2. (2 pts each) True or False? You needn't justify your answers.

(a) $\mathbb{R}^{2\times 2}$ (the space of 2×2 matrices) is isomorphic to P_3 (the space of polynomials of degree at most 3). (That is, there exists an invertible linear transformation between them.)

TRUE. They are both four-dimensional. In particular, if we choose a basis for each of them, we see that they are each isomorphic to $\mathbb{R}^4$ (by the coordinates in the chosen bases).

(b) If $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is a basis for a linear space V, then $(\vec{v}_2, \vec{v}_2 - \vec{v}_1, \vec{v}_3)$ is also a basis for V.

TRUE. We can see that the latter list of vectors has the same span as the former because each of v_1, v_2, and v_3 are in its span (for example, $v_1 = v_2 - (v_2 - v_1)$). Thus the latter list is a spanning set of minimal size (because we can't have a spanning set smaller than a basis), so it's linearly independent as well (i.e. there are no redundant vectors), so it is a basis.