QUIZ SOLUTIONS #10, 9/27/07

MATH 54, FALL 2007

Show your work and justify your answers! Feel free to use both sides.

Name:

1. (4 pts) Write the transformation given in the standard basis by the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ in terms of the basis $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

We let $S = \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}$ be the change-of-basis matrix. Then $S^{-1}AS$ is the matrix in the $(\vec{v}_1, \vec{v}_2)$ basis. Thus we need to find S^{-1}:

$$\left[\begin{array}{cc|cc} -1 & -2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \times -1 \quad \leftrightarrow \quad \left[\begin{array}{cc|cc} 1 & 2 & -1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] -I \quad \leftrightarrow$$
$$\left[\begin{array}{cc|cc} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] -2II \quad \leftrightarrow \quad \left[\begin{array}{cc|cc} 1 & 0 & -3 & -2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

Thus $S^{-1} = \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix}$. Thus our answer is

$$\begin{aligned} S^{-1}AS &= \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -8 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

2. (a) (3 pts) Show (by checking the conditions) that the following is a subspace: the space of all 2×2 matrices with $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in their kernel.

Let V be the space of 2×2 matrices with $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in their kernel.
We check the conditions for this to be a subspace (of the space of all 2×2 matrices):

- Zero is in V: yes, the kernel of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is all of $\mathbb{R}^2$, and in particular contains the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- Closed under scalar multiplication: Suppose A is in V. We show kA is in V (for k any scalar). We calculate: $(kA)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = k\left(A\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = k(\overrightarrow{0}) = \overrightarrow{0}$, so indeed kA is in V.
- Closed under addition: Suppose A and B are in V. We show $A+B$ is in V. We calculate: $(A+B)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = A\begin{bmatrix} 0 \\ 1 \end{bmatrix} + B\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \overrightarrow{0} + \overrightarrow{0} = \overrightarrow{0}$, so indeed $A+B$ is in V.

Thus V is a subspace of the space of all 2×2 matrices.

(b) (3 pts) Find a basis for the subspace given in part (a).

First we check what it means for a matrix to be in V. Write a general 2×2 matrix A as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A is in V means $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \overrightarrow{0}$. We calculate: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$. Thus A is in V if and only if $b = 0$ and $d = 0$. Thus the general form for a matrix in V is $\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$. Thus a basis for V is:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$