Worksheet 5

1. False, e.g. if $P = x^6$ then $P^{(6)} = 6\cdot5\cdot4\cdot3\cdot2$

We can fix it by saying that $p^{(7)} = 0$.

2. By the chain rule: $\left(\sin(\sin x)\right)' = \cos(\sin x)\cdot\cos x$

3. $f(x) = \sin(2x)$

$f'(x) = 2\cos(2x)$

$f''(x) = -4\sin(2x)$

$f'''(x) = -2^3\cos(2x)$

$f^{(4)}(x) = 2^4\sin 2x$

Notice the pattern. It alternates between sin, cos and $+, +, -, -$.

So $\boxed{f^{(100)} = 2^{100}\sin 2x}$

4. • $\frac{d}{dx}\left(\ln(\tan x + x^2)\right) = \frac{1}{\tan x + x^2}\cdot(\sec^2 x + 2x)$

• $\frac{d}{dx}\left(3^{x^2}\right) = 3^{x^2}\cdot\ln 3\cdot 2x$

• Let $y = \sqrt[4]{\frac{x^2+1}{x^2-1}}$. Then $\ln y = \ln\sqrt[4]{\frac{x^2+1}{x^2-1}}$

so, $\ln y = \frac{1}{4}\ln\frac{x^2+1}{x^2-1} = \frac{1}{4}\left(\ln(x^2+1) - \ln(x^2-1)\right)$

So $\frac{y'}{y} = \frac{1}{4}\left(\frac{2x}{x^2+1} - \frac{2x}{x^2-1}\right)$

So $y' = \frac{1}{4}\sqrt[4]{\frac{x^2+1}{x^2-1}}\cdot\left(\frac{2x}{x^2+1} - \frac{2x}{x^2-1}\right)$

2. $\sqrt{x} + \sqrt{y} = 1$

apply $\frac{d}{dx}$: $\frac{1}{2} x^{-1/2} + \frac{1}{2} y^{-1/2} \cdot y' = 0$

apply it again $-\frac{1}{4} x^{-3/2} - \frac{1}{4} y^{-3/2} (y') \cdot (y') + \frac{1}{2} y^{-1/2} y'' = 0$

So $\frac{1}{2} y^{-1/2} y'' = \frac{1}{4} x^{-3/2} + \frac{1}{4} y^{-3/2} (y')^2$

$y'' = \frac{1}{2} x^{-3/2} y^{1/2} + \frac{1}{4} y^{-1/2} (y')^2$

3. $x^2 + y^2 = 25$

apply $\frac{d}{dx}$ = $2x + 2y \cdot y' = 0$

$\boxed{y' = \frac{-x}{y}}$ at $(3,4)$, $\boxed{y' = \frac{-3}{4}}$

$$\frac{d^2y}{dx^2} = y'' = \frac{-1 \cdot y + y' \cdot x}{y^2} = \frac{-1 \cdot y + \left(\frac{-x}{y}\right) x}{y^2}$$

$$= \frac{-y^2 - x^2}{y^3}$$

4. $\lim_{x \to \infty} x \sin \frac{1}{x} = \lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{y \to 0} \frac{\sin y}{y} = 1.$

if $y = \frac{1}{x}$

5. $y^3 + xy + x^4 = 11.$

Apply $\frac{d}{dx}$:

$$3y^2 \cdot y' + (x)' \cdot y + x \cdot (y)' + 4x^3 = 0$$

$$3y^2 \cdot y' + 1 \cdot y + xy' + 4x^3 = 0$$

$$y'(3y^2 + x) + y + 4x^3 = 0$$

Apply $\frac{d}{dx}$ again:

$$y''(3y^2 + x) + (y')(6y \cdot y' + 1) + y' + 12x^2 = 0$$

So

$$y'' = \frac{-12x^2 - y' - y'(6y \cdot y' + 1)}{3y^2 + x}$$

now plug in $y' = \dfrac{-y - 4x^3}{3y^2 + x}$

to get

$$\boxed{y'' = \frac{-12x^2 - \left(\frac{-y-4x^3}{3y^2+x}\right) - \left(\frac{-y-4x^3}{3y^2+x}\right)\left(6y \cdot \left(\frac{-y-4x^3}{3y^2+x}\right) + 1\right)}{3y^2 + x}}$$