Math 1A Sections 308-309
Worksheet 10: November 16,2009

Warm-up Questions - Work by yourself or with a neighbor

1. Find all functions $f(x)$ with $f''(x) = e^x$.

   Solution: $f(x) = e^x + Cx + D$.

2. Find a function $f(x)$ such that $f''(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f. (This is a harder problem, but don't give up - Draw a picture, think about what $f(x)$, $f'(x)$ are. I know everyone can do this!)

   Solution: We take some antiderivatives to find that $f'(x) = 1/4x^4 + C$ and $f(x) = 1/20x^5 + Cx + D$. To solve this problem we have to pick a point on the graph where the function is supposed to be tangent. Any point will do, so let's pick the easiest one - $(0, 0)$. In this case since our function is supposed to go through $(0, 0)$ we need $f(0) = 0$ and because the line $y = -x$ is supposed to be tangent at this point, we must have $f'(0) = -1$. Using the formulas for $f(x)$ and $f'(x)$ above, we see that $C = -1$ and $D = 0$, so $f(x) = 1/20x^5 - x$.

Group Problems

1. Find antiderivatives for the following functions. Don't use anything fancy like u-substitution. See if you can "guess" by thinking about the chain rule, product rule, and quotient rule.

   - $f(x) = \sin x + x\cos x$, Solution: $x \sin x$. (product rule)
   - $f(x) = 14x^5$, Solution: $14/6x^6$
   - $f(x) = 2x\cos(x^2)$ Solution: $\sin(x^2)$ (chain rule)
   - $f(x) = \cosh(x)$. Solution: $\sinh(x)$.

2. Estimate the area under the graph of the function $y = x^2$ from $x = 0$ to $x = 4$ by using 4 rectangles (endpoints on the right). Can you write down a formula for the area if we use n rectangles? Use the Σ notation if you know it. Can you evaluate this sum?

   Solution: If we use 4 rectangles then the rectangles all have width 1, and heights of $1, 4, 9, 16$ (the y-values when $x = 1, 2, 3, 4$. So

   $$A = 1 \cdot 1 + 1 \cdot 4 + 1 \cdot 9 + 1 \cdot 16 = 30.$$

With n rectangles we have that the width is $4/n$ (since we have to divide the interval from 0 to 4 into n pieces. The heights are $f(4/n)$, $f(8/n)$ etc, So then the area is

$$A = 4/n \cdot (4/n)^2 + 4/n \cdot (8/n)^2 + \cdots + (4n/n)^2 = \sum_{i=1}^{n} 4/n(4i/n)^2$$

(Come talk in office hours and we can go over this in detail).

3. Let $f(x) = x \cos x$. Write down a formula for the area under the graph of $f(x)$ from $x = 0$ to $x = 1$ as a limit of a summation.

   Solution:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} (1/n) \left[(i/n) \cos(i/n) \right].$$

4. Can you write down some functions such that $f''(x) = f'''(x)$? Can you write down the general form of all such functions?

   Solution: We start with the equation $f''(x) - f'''(x) = 0$. Taking the antiderivative of both sides yields:

$$f'(x) - f''(x) = C.$$

$$f(x) - f'(x) = Cx + D.$$

   This is probably as far as you're able to get now, but once you know a bit more, you can actually say that the solutions are all functions $f(x)$ of the form:

$$f(x) = (Cx + D) + e^x$$