THE HOMOLOGY OF $BU\langle 2k\rangle$

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1.1. What are $BU\langle 2k\rangle$ and $MU\langle 2k\rangle$? A primary use of topology is to speak of parametrized families of objects, called bundles. One of the characterizing properties of algebraic topology and homotopy theory is the recurring theme of representability, which appears in this context too: for many types C of bundles there exists a classifying space $B\mathsf{C}$ so that an isomorphism class of C-bundles over Y corresponds to a homotopy class of maps $f : Y \to B\mathsf{C}$. The cohomology of these spaces $B\mathsf{C}$ is then of great interest, since a class $x \in H^*(B\mathsf{C}; E)$ pulls back to a class $f^*x \in H^*(Y; E)$. This gives us two kinds of information: plainly, if two C-bundles have different characteristic classes, then they must be nonisomorphic, since homotopic classifying maps induce the same map on cohomology. More interestingly, characteristic classes frequently also have geometric interpretations, though this is not a perspective that admits a uniform, structured treatment.

For example, the classifying space for (stable, almost-complex) vector bundles is denoted $BU\langle 0\rangle$. Its cohomology, a classical calculation[1], is polynomial with a new generator in every even degree $2n$, called the nth Chern class:[2]

$$H^*(BU\langle 0\rangle; \mathbb{Z}) = \mathbb{Z}[c_0, c_0^{-1}][c_1, c_2, c_3, \ldots].$$

When characteristic classes vanish, we find ourselves in the uncomfortable position of losing what was potentially valuable geometric information about our family; we describe a way of circumventing this for particular characteristic classes of complex vector bundles. Consider the following Whitehead-Postnikov tower, which describes a decomposition of $BU\langle 0\rangle$ as being built out of fiber sequences over Eilenberg-Mac Lane spaces:

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
 & & BU\langle 6\rangle & \longrightarrow & K(\mathbb{Z}, 6) \\
 & & \downarrow & & \\
BSU & = & BU\langle 4\rangle & \longrightarrow & K(\mathbb{Z}, 4) \\
 & & \downarrow & & \\
BU & = & BU\langle 2\rangle & \longrightarrow & K(\mathbb{Z}, 2) \\
 & & \downarrow & & \\
BU \times \mathbb{Z} & = & BU\langle 0\rangle & \longrightarrow & K(\mathbb{Z}, 0).
\end{array}$$

Each corner $BU\langle 2k+2\rangle \to BU\langle 2k\rangle \to K(\pi_{2k} BU\langle 0\rangle, 2k) = K(\mathbb{Z}, 2k)$ is a fiber sequence. Now, integral cohomology also enjoys the property of representability, and each of these maps $BU\langle 2k\rangle \to K(\mathbb{Z}, 2k)$ is selecting a particular cohomology class in $x_{2k} \in H^{2k}(BU\langle 2k\rangle; \mathbb{Z})$, i.e., a characteristic class. Given a classifying map $f : Y \to BU\langle 2k\rangle$ for a family on Y, if the characteristic class f^*x_{2k} vanishes in $H^*(Y; \mathbb{Z})$ then this corresponds to the composite $Y \to BU\langle 2k\rangle \to K(\mathbb{Z}, 2k)$ being nullhomotopic. This means that f lifts

[1] The not-so-classical version is $BU\langle 2\rangle_{H\mathbb{Z}} = \mathrm{Div}^+(\mathbb{C}\mathrm{P}^\infty_{H\mathbb{Z}})$, which takes into account that the Chern classes arise as certain symmetric polynomials of the classes in $H^*BU(n)$.

[2] The c_0, c_0^{-1} business has to do with the splitting $BU\langle 0\rangle = \mathbb{Z} \times BU\langle 2\rangle$, where $BU\langle 2\rangle$ is simply connected.

through the fiber to a map $\tilde{f} : Y \to BU\langle 2k+2\rangle$, and so the cohomology $H^*(BU\langle 2k\rangle; E)$ gives us a rich new theory of characteristic classes associated to Y previously not visible.[3]

The cohomologies $H^*(BU\langle 4\rangle; \mathbb{Z})$ and $H^*(BU\langle 6\rangle; \mathbb{Z}/2)$ are accessible by the Serre spectral sequence and the Kudo transgression theorem, using the fibrations $K(\mathbb{Z}, 1) \to BU\langle 4\rangle \to BU\langle 2\rangle$ and $K(\mathbb{Z}, 3) \to BU\langle 6\rangle \to BU\langle 4\rangle$. Namely, we find

$$H^*(BU\langle 4\rangle; \mathbb{Z}) = H^*(BU\langle 2\rangle; \mathbb{Z})/c_1,$$
$$H^*(BU\langle 6\rangle; \mathbb{Z}/2) = \frac{H^*(BU\langle 4\rangle; \mathbb{Z})}{[c_2, c_3, c_5, \ldots, c_{2^i+1}, \ldots]} \otimes \mathrm{Op}[\mathrm{Sq}^3 \iota_3].$$

In the spectral sequence for $BU\langle 6\rangle$, ι_3 kills c_2 by a transgression, and hence $\mathrm{Sq}^I \iota_3$ kills c_3, c_5, and so on (mod decomposables) for $\mathrm{Sq}^2 \iota_3$, $\mathrm{Sq}^4 \mathrm{Sq}^2 \iota_3$, and so on.[4] Apart from these differentials, however, the spectral sequence collapses, and so in particular classes of the form $\iota_3^2 = \mathrm{Sq}^3 \iota_3$ survive — indeed, everything of the form $\mathrm{Sq}^I \mathrm{Sq}^3 \iota_3$ survives. These are the new characteristic classes we were seeking.

1.2. **Singer's calculation of $H^*(BU\langle 2k\rangle; \mathbb{F}_p)$.** These algebras have been computed in general, first by Bob Stong with $\mathbb{F}_2$ coefficients and then by William Singer with $\mathbb{F}_p$ coefficients. Singer's formula is:

$$H^*(BU\langle 2k\rangle; \mathbb{Z}/p) = \frac{H^*(BU; \mathbb{Z}/p)}{\mathbb{Z}/p[\theta_{2i} | \sigma_p(i-1) < k-1]} \otimes \prod_{t=0}^{p-2} F[M_{2k-3-2t}],$$

where

- Over $\mathbb{Z}/p$, $F[M_n]$ is defined to be $\mathrm{Op}[\beta P^1 \iota_n]$, the Steenrod subalgebra of $H^*(K(\mathbb{Z}, n); \mathbb{Z}/p)$ generated by $\beta P^1 \iota_n$. Over $\mathbb{Z}/2$, $F[M_n]$ is defined to be $\mathrm{Op}[\mathrm{Sq}^3 \iota_n]$, the Steenrod subalgebra of $H^*(K(\mathbb{Z}, n); \mathbb{Z}/2)$ generated by $\mathrm{Sq}^3 \iota_n$.
- $\sigma_p(n)$ is the $\mathbb{N}$-valued p-adic digital sum of n. So, for instance, $\sigma_2(4) = \sigma_2(1 \cdot 2^2) = 1$, $\sigma_2(3) = \sigma_2(1 \cdot 2^0 + 1 \cdot 2^1) = 2$, and $\sigma_3(11) = \sigma_3(2 \cdot 3^0 + 1 \cdot 3^2) = 3$.
- θ_{2i} is equal to c_{2i} mod decomposables.

Singer accomplished this computation using the Eilenberg-Moore spectral sequence applied to a fiber square

$$\begin{array}{ccc} K(\mathbb{Z}, 2k-1) & \longrightarrow & BU\langle 2k+2\rangle \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & BU\langle 2k\rangle, \end{array}$$

which is a spectral sequence of the form

$$\mathrm{Tor}_{*,*}^{H^*(BU\langle 2k\rangle; \mathbb{Z}/p)}(H^*(\mathrm{pt}; \mathbb{Z}/p), H^*(BU\langle 2k+2\rangle; \mathbb{Z}/p)) \Rightarrow H^*(K(\mathbb{Z}, 2k-3); \mathbb{Z}/p).$$

This is a pretty gross ring. But, at least when $k \leq 3$, it is even-concentrated, and so we can attempt a functorial description of $\mathrm{Spf}\, H^*(BU\langle 2k\rangle; \mathbb{Z}/p) = BU\langle 2k\rangle_{H\mathbb{Z}/p}$, our favorite thing to do. Indeed, because $BU\langle 2k\rangle$ is an even-concentrated H-space in this range, we might also find a description of $\mathrm{Spec}\, H_*(BU\langle 2k\rangle; \mathbb{Z}/p) = BU\langle 2k\rangle^{H\mathbb{Z}/p}$, as these two schemes are related by the formula $\underline{\mathrm{Hom}}(BU\langle 2k\rangle_{H\mathbb{Z}/p}, \hat{\mathbb{G}}_m) \cong BU\langle 2k\rangle^{H\mathbb{Z}/p}$, i.e., through Cartier duality, the formal scheme analogue of Pontryagin duality.

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2.1. **Constructing C^k.** To study these spaces $BU\langle 2k\rangle$, we should begin by constructing at least one complex vector bundle V_k over a space X_k whose classifying map to $BU\langle 0\rangle$ lifts to $BU\langle 2k\rangle$. Our first thing to note is that the spaces $BU\langle 2k\rangle$ assemble into a ring spectrum called connective K-theory, where $ku^{2n}(X) =$

[3] At low orders, these classifying spaces are sometimes also given the following names: $BU\langle 4\rangle$, $BO\langle 2\rangle$, and $BO\langle 3\rangle$ can be identified with the classifying spaces of Lie groups BSU, BO, and $BSpin$ respectively. The space $BO\langle 8\rangle$ is also called $BString$, though $String$ is not a compact Lie group.

[4] This uses the Wu formula $\mathrm{Sq}^k c_m = \sum_{i=0}^k \binom{k-m}{i} c_{k-i} c_{m+i}$.

$\pi_0\mathsf{Spaces}(X, BU\langle 2k\rangle)$ and the multiplicative structure is given by the lifting of tensor product from KU.[5] Then, we can produce an element $\mathcal{L} - 1 \in ku^2(\mathbb{C}\mathrm{P}^\infty)$, since $\mathcal{L} = (\mathbb{C}^\infty \setminus \{0\} \downarrow \mathbb{C}\mathrm{P}^\infty)$ is a complex line bundle of rank 1 and $\mathcal{L} - 1$ is normalized to have virtual rank 0. Then, we use the external product on ku to build a class $(\mathcal{L} - 1)^{\otimes k} \in ku^{2k}((\mathbb{C}\mathrm{P}^\infty)^{\times k})$. This gives a preferred map

$$f_k : (\mathbb{C}\mathrm{P}^\infty)^k \to BU\langle 2k\rangle.$$

On homology, this induces a map

$$H_*(f_k; E) : H_*((\mathbb{C}\mathrm{P}^\infty)^{\times k}; E) \to H_*(BU\langle 2k\rangle; E),$$

which by a universal coefficient theorem corresponds to an element

$$f'_k \in H^*((\mathbb{C}\mathrm{P}^\infty)^k; E)\hat{\otimes} H_*(BU\langle 2k\rangle; E) = H_*(BU\langle 2k\rangle; E)[\![x_1, \ldots, x_k]\!].$$

Because this power series was induced by the product bundle $(\mathcal{L} - 1)^{\otimes k}$, it enjoys various properties, including the following:

- Rigidity: $f(x_1, \ldots, 0, \ldots, x_k) = 1$. This comes from $(\mathcal{L} - 1) \otimes \cdots \otimes 0 \otimes \cdots \otimes (\mathcal{L} - 1) = 0$.
- Symmetry: $f(x) = f(\sigma x)$. This comes from $\mathcal{L} \otimes \mathcal{L}' = \mathcal{L}' \otimes \mathcal{L}$.
- 2-cocycle: $f(x_1, \ldots, x_k)f(x_0 +_E x_1, ..., x_k)^{-1}f(x_0, x_1 +_E x_2, \ldots, x_k)f(x_0, x_1, x_3, ..., x_k)^{-1} = 1$. We want to interrelate this power series we're producing with the formal group $\mathbb{C}\mathrm{P}^\infty_E$, and here's our opportunity to do so. Define the following maps $(\mathbb{C}\mathrm{P}^\infty)^{k+1} \to (\mathbb{C}\mathrm{P}^\infty)^k$:
  - π_i, given by dropping the ith copy of $\mathbb{C}\mathrm{P}^\infty$.
  - m_i, given by applying the multiplication map $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$ to the ith and $(i + 1)$th components, corresponding to tensor product of line bundles.

  We can then compute the pullback bundles along these maps:

$$\pi_s^*(\mathcal{L} - 1)^{\otimes k} = \bigotimes_{\substack{1\le i\le k+1\\ i\ne s}} (\mathcal{L}_i - 1), \qquad m_s^*(\mathcal{L} - 1)^{\otimes k} = (\mathcal{L}_s\mathcal{L}_{s+1} - 1) \cdot \bigotimes_{\substack{1\le i\le k+1\\ i\ne s,s+1}} (\mathcal{L}_i - 1).$$

  Next, we make note of the following isomorphism (where $s \neq t$):

$$\begin{aligned}(m_s^* - \pi_s^* - \pi_{s+1}^*)(\mathcal{L} - 1)^{\otimes k} &= \left((\mathcal{L}_s\mathcal{L}_{s+1} - 1) \cdot \bigotimes_{\substack{1\le i\le k+1\\ i\ne s,s+1}} (\mathcal{L}_i - 1) - \bigotimes_{\substack{1\le i\le k+1\\ i\ne s}} (\mathcal{L}_i - 1) - \bigotimes_{\substack{1\le i\le k+1\\ i\ne s+1}} (\mathcal{L}_i - 1)\right) \\ &= ((\mathcal{L}_s\mathcal{L}_{s=1} - 1) - (\mathcal{L}_{s+1} - 1) - (\mathcal{L}_s - 1)) \bigotimes_{\substack{1\le i\le k+1\\ i\ne s,s+1}} (\mathcal{L}_i - 1) \\ &= \bigotimes_{1\le i\le k+1} (\mathcal{L}_i - 1).\end{aligned}$$

  In particular, this means the following isomorphisms hold for all choices of s and t:

$$(m_s^* - \pi_s^* - \pi_{s+1}^*)(\mathcal{L} - 1)^{\otimes k} \cong (m_t^* - \pi_t^* - \pi_{t+1}^*)(\mathcal{L} - 1)^{\otimes k}.$$

  Selecting $s = 1$ and $t = 2$ gives the following identity in terms of our power series f'_k:

$$\frac{f'_k(x_1, \ldots, x_k)}{f'_k(x_0 +_E x_1, x_2, \ldots, x_k)} \cdot \frac{f'_k(x_0, x_1 +_E x_2, x_3, \ldots, x_k)}{f'_k(x_0, x_1, x_3, \ldots, x_k)} = 1.$$

  We call this the multiplicative 2-cocycle condition in k variables.

One can construct a scheme $C^k(\mathbb{C}\mathrm{P}^\infty_E; \hat{\mathbb{G}}_m)$ of all such k-variate symmetric rigid 2-cocycles, and this construction shows that we get a classifying map

$$BU\langle 2k\rangle^E \xrightarrow{f'_k} C^k(\mathbb{C}\mathrm{P}^\infty_E; \hat{\mathbb{G}}_m).$$

[5]At the very least, we can define connective K-theory as the nonnegative truncation of KU: $kU = \tau_{\ge 0}KU$. Then, because $\tau_{\ge 0}$ is right-adjoint to the inclusion of connective spectra into all spectra, it preserves preserves products and hence ring spectra to ring spectra. That the spaces representing kU match up as claimed is more complicated, and uses special properties of BU — it's not true, for instance, for $\tau_{\ge 0}KO$.

Our goal now is to study the size of the target and the behavior of the classifying map; our analysis will show that this turns out to actually be quite good, with an isomorphism for $k \leq 3$ and with a seriously strong relation for $k > 3$.

2.2. **Reductions.** Suppose E is an even-periodic ring spectrum.[6] We want to study $H_*(BU\langle 2k\rangle; E)$ and $C^k(\mathbb{CP}^\infty_E; \hat{\mathbb{G}}_m)$. Because the homology of $BU\langle 2k\rangle$ is even-concentrated in the cases $k \leq 3$ that we computed, the Atiyah-Hirzebruch spectral sequence

$$H_p(BU\langle 2k\rangle; H_q(\mathrm{pt}; E)) \Rightarrow H_{p+q}(BU\langle 2k\rangle; E)$$

collapses at E^2. Hence, we reduce to studying the case of the ring spectrum HR for various coefficient rings R, whose associated formal group $\mathbb{CP}^\infty_{HR}$ is the additive formal group $\hat{\mathbb{G}}_a$. We're thus interested in the space of power series $f(x_1, \ldots, x_k)$ satisfying the above three conditions, with $x +_E y$ replaced with just $x + y$. The Hasse local-to-global principle tells us that it's enough to study $H_* BU\langle 2k\rangle$ and $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$ with $\mathbb{Z}_{(p)}$-coefficients for $p = 0, 2, 3, 5, \ldots$.

This reduces one of the formal groups in $C^k(\mathbb{CP}^\infty_E; \hat{\mathbb{G}}_m)$ to the manageable formal group $\hat{\mathbb{G}}_a$; now we seek to reduce some of the complexities introduced by $\hat{\mathbb{G}}_m$. Formal groups are supposed to behave very similarly to infinitesimal Lie groups, and so we take the following cue from Lie theory: if you want to understand a Lie group, you start by understanding its tangent space at the identity. The tangent bundle of a manifold consists, intuitively, of points of the manifold equipped with tangent directions, specified perhaps by the linear part of a curve passing through that point. We can make a similar definition in the context of formal schemes: for a formal scheme X, the Zariski tangent bundle TX is defined to be

$$TX = \underline{\mathrm{Hom}}(\operatorname{Spec} k[\varepsilon]/\varepsilon^2, X).$$

Computationally, this scheme has the property that

$$\begin{aligned} TX(R) &= \mathrm{Hom}(\operatorname{Spec} R, \underline{\mathrm{Hom}}(\operatorname{Spec} k[\varepsilon]/\varepsilon^2, X)) \\ &= \mathrm{Hom}(\operatorname{Spec} R \times \operatorname{Spec} k[\varepsilon]/\varepsilon^2, X) \\ &= \mathrm{Hom}(\operatorname{Spec} R[\varepsilon]/\varepsilon^2, X) \\ &= X(R[\varepsilon]/\varepsilon^2). \end{aligned}$$

This scheme TX maps down onto the scheme $\underline{\mathrm{Hom}}(\operatorname{Spec} k, X)$ of k-valued points of X, and we also have a map $CX \to TX$ from curves in X to the tangent bundle of X induced by the map $\hat{\mathbb{A}}^1 \leftarrow \operatorname{Spec} k[\varepsilon]/\varepsilon^2$, which is in turn induced by the quotient $k[\![x]\!] \to k[\varepsilon]/\varepsilon^2$. We have a preferred k-point $\eta : \operatorname{Spec} k \to \mathbb{CP}^\infty_E$ given by the identity element of the formal group $\mathbb{CP}^\infty_E$. The pullback of $T\mathbb{CP}^\infty_E$ along η is denoted $T_1\mathbb{CP}^\infty_E$ for the tangent space at the identity.

The scheme $T_1\mathbb{CP}^\infty_E$ is also a formal group, and it turns out to be isomorphic to $\hat{\mathbb{G}}_a$, which one can see by express computation: let $F(x, y) = x + y + xyf(x, y)$ be the formal group law associated to $\mathbb{CP}^\infty_E$ after picking a coordinate. Then for two points $u, v \in T_1\mathbb{CP}^\infty_E(R)$, we compute their sum to be

$$u +_{T_1\mathbb{CP}^\infty_E} v = \varepsilon u +_{\mathbb{CP}^\infty_E} \varepsilon v = \varepsilon u + \varepsilon v + \varepsilon^2 uvf(\varepsilon u, \varepsilon v) = \varepsilon u +_{\hat{\mathbb{G}}_a} \varepsilon v.$$

Similarly, one can compute that $T_1 C^k(F; G) = C^k(F; T_1 G)$, and hence $T_1 C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m) = C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a)$. For arbitrary ground rings and k, this[7] will be our first object of study, and we'll try to recover $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$ (and, hence, $C^k(\mathbb{CP}^\infty_E; \hat{\mathbb{G}}_m)$) from it later on.

3. 4/25: Calculation of $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a)$

[8]

As ever, the 0-local case is easier than the p-local case for $p > 0$, so we'll deal with it first. We can certainly produce an element of $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a) \times \operatorname{Spec} \mathbb{Q}$ as follows: the 2-cocycle condition is named as such because it

[6] So, E is automatically complex orientable.

[7] This scheme has actually been studied before; for instance, $C^2(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a)$ plays a prominent role in Lazard's symmetric 2-cocycle lemma.

[8] This calculation is due variously to Lazard for $k = 2$ in 1955, Ando, Hopkins, and Strickland for $k = 3$ sometime in 1995-2001, and to Hughes, Lau, and myself for $k > 3$ in 2008.

corresponds to requesting that $u : F^2 \to G$ vanish under a boundary map given by building the simplicial object BF, Hom-ing into G, and taking the cochain complex associated to that cosimplicial object. Hence, if we have a map $s : F \to G$, we can apply the previous coboundary operator $\delta^1(s) = s(x+y) - s(x) - s(y)$ to produce a 2-cocycle. Writing (I) for the symmetric polynomial

$$(I) = \sum_{\sigma \in \Sigma_{\ell(I)}} \mathbf{x}^{\sigma I},$$

this is exactly what we do:

$$\zeta_k^n = d^{-1}(\delta^1)^{\circ(k-1)} x^n = d^{-1} \sum_{\emptyset \neq X \subset \{x_1, \ldots, x_k\}} (-1)^{|X|} \left(\sum_{x \in X} x \right)^n = d^{-1} \sum_{0 \notin \lambda} \binom{n}{\lambda} (\lambda),$$

where $d = \gcd_{0 \notin \lambda} \binom{n}{\lambda}$ is the largest positive integer that leaves the thing integral. Rationally, this is in fact the only thing that happens; the proof of this fact is not hard, but it's also not interesting, so we will skip it.[9]

The p-local case is much more interesting. The case of $\mathbb{Z}_{(p)}$ can be studied through successive approximation by the rings $\mathbb{Z}_{(p)}/p^j$, beginning with $\mathbb{Z}_{(p)}/p = \mathbb{F}_p$. It's much more satisfying to see this analysis with some data in front of you, so here are two gigantic tables of 2-cocycles over $\mathbb{F}_2$ and $\mathbb{F}_3$, computed by brute force with Mathematica's `NullSpace[]` function:

	dim 2	3	4	5	6
2	$(1,1)$	0	0	0	0
3	$(2,1)$	$(1,1,1)$	0	0	0
4	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$	0	0
5	$(4,1)$	$(2,2,1)$	$(2,1,1,1)$	$(1,1,1,1,1)$	0
6	$(4,2)$	$(2,2,2)$, $(4,1,1)$	$(2,2,1,1)$	$(2,1,1,1,1)$	$(1,1,1,1,1,1)$
7	$(6,1)+$ $(5,2)+$ $(4,3)$	$(4,2,1)$	$(2,2,2,1)$, $(4,1,1,1)$	$(2,2,1,1,1)$	$(2,1,1,1,1,1)$
8	$(4,4)$	$(4,2,2)$	$(2,2,2,2)$, $(4,2,1,1)$	$(2,2,2,1,1)$, $(4,1,1,1,1)$	$(2,2,1,1,1,1)$
9	$(8,1)$	$(4,4,1)$	$(4,2,2,1)$	$(2,2,2,2,1)$, $(4,2,1,1,1)$	$(2,2,2,1,1,1)$, $(4,1,1,1,1,1)$
10	$(8,2)$	$(4,4,2)$, $(8,1,1)$	$(4,2,2,2)$, $(4,4,1,1)$	$(2,2,2,2,2)$, $(4,2,2,1,1)$	$(2,2,2,2,1,1)$, $(4,2,1,1,1,1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

FIGURE 1. Cocycles with coefficients in $\mathbb{F}_2$.

One pattern immediately visible in the tables is that there are a lot of powers of p, which is also immediately explainable: since $(x +_{\hat{\mathbb{G}}_a} y)^{p^j} = x^{p^j} +_{\hat{\mathbb{G}}_a} y^{p^j}$ in characteristic p, all symmetric polynomials of the form (I) with I consisting of powers of p are immediately 2-cocycles. This explains the presence of nearly all the cocycles in the $\mathbb{F}_2$ table, but there's a whole lot of other things happening in the $\mathbb{F}_3$ table, particularly where no power-of-3 indices exist — for instance, the space of cocycles in degree 12 and dimension 3 with $\mathbb{F}_3$ coefficients is generated by $(6,3,3)$ and $(9,2,1) - (10,1,1)$.

[9]See Ando, Hopkins, Strickland's proposition A.1. The main idea is that this composite $(\delta^1)^{\circ(k-1)}$ has an inverse given by solving a particular ODE using series methods.

	dim 2	3	4	5
deg 2	$(1,1)$	0	0	0
3	$(2,1)$	$(1,1,1)$	0	0
4	$(3,1)$	$(2,1,1)$	$(1,1,1,1)$	0
5	$(3,2)-$ $(4,1)$	$(3,1,1)$	$(2,1,1,1)$	$(1,1,1,1,1)$
6	$(3,3)$	$(3,2,1)-$ $(4,1,1)$	$(3,1,1,1)$	$(2,1,1,1,1)$
7	$(4,3)-$ $(6,1)$	$(3,3,1)$	$(3,2,1,1)-$ $(4,1,1,1)$	$(3,1,1,1,1)$
8	$(6,2)+$ $(4,4)-$ $(7,1)-$ $(5,3)$	$(6,1,1)-$ $(4,3,1)+$ $(3,3,2)$	$(3,3,1,1)$	$(3,2,1,1,1)-$ $(4,1,1,1,1)$
9	$(6,3)$	$(3,3,3)$	$(6,1,1,1)-$ $(4,3,1,1)+$ $(3,3,2,1)$	$(3,3,1,1,1)$
10	$(9,1)$	$(4,3,3)-$ $(6,3,1)$	$(3,3,3,1)$	$(6,1,1,1,1)-$ $(4,3,1,1,1)+$ $(3,3,2,1,1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

FIGURE 2. Cocycles with coefficients in $\mathbb{F}_3$.

We do still have one tool available: the cocycle ζ_k^n over $\mathbb{Z}$ is also a cocycle over $\mathbb{F}_p$ after pushing forward the coefficients. For instance, we calculate

$$\begin{aligned}\zeta_3^{12} = {} & 2x^{10}yz + 10x^9y^2z + 30x^8y^3z + 60x^7y^4z + 84x^6y^5z + 84x^5y^6z + 60x^4y^7z + 30x^3y^8z + 10x^2y^9z + \\ & + 2xy^{10}z + 10x^9yz^2 + 45x^8y^2z^2 + 120x^7y^3z^2 + 210x^6y^4z^2 + 252x^5y^5z^2 + 210x^4y^6z^2 + 120x^3y^7z^2 + \\ & + 45x^2y^8z^2 + 10xy^9z^2 + 30x^8yz^3 + 120x^7y^2z^3 + 280x^6y^3z^3 + 420x^5y^4z^3 + 420x^4y^5z^3 + 280x^3y^6z^3 + \\ & + 120x^2y^7z^3 + 30xy^8z^3 + 60x^7yz^4 + 210x^6y^2z^4 + 420x^5y^3z^4 + 525x^4y^4z^4 + 420x^3y^5z^4 + 210x^2y^6z^4 + \\ & + 60xy^7z^4 + 84x^6yz^5 + 252x^5y^2z^5 + 420x^4y^3z^5 + 420x^3y^4z^5 + 252x^2y^5z^5 + 84xy^6z^5 + 84x^5yz^6 + \\ & + 210x^4y^2z^6 + 280x^3y^3z^6 + 210x^2y^4z^6 + 84xy^5z^6 + 60x^4yz^7 + 120x^3y^2z^7 + 120x^2y^3z^7 + 60xy^4z^7 + \\ & + 30x^3yz^8 + 45x^2y^2z^8 + 30xy^3z^8 + 10x^2yz^9 + 10xy^2z^9 + 2xyz^{10},\end{aligned}$$

and hence

$$\begin{aligned}\zeta_3^{12} & \equiv 2x^{10}yz + x^9y^2z + x^2y^9z + 2xy^{10}z + x^9yz^2 + xy^9z^2 + x^6y^3z^3 + x^3y^6z^3 + x^3y^3z^6 + x^2yz^9 + xy^2z^9 + 2xyz^{10} \\ & \equiv (6,3,3) + (9,2,1) - (10,1,1) \mod 3.\end{aligned}$$

This is no accident.

Here's the key observation that will make all our combinatorics tick: a positive integer n can be written as a sum of powers of p: $n = p^{i_1} + p^{i_2} + \cdots$, which in turn tells us how $(x+y)^n$ expands in characteristic p:

$$(x+y)^n = (x+y)^{p^{i_1}+p^{i_2}+\cdots} = (x+y)^{p^{i_1}}(x+y)^{p^{i_2}}\cdots = (x^{p^{i_1}} + y^{p^{i_1}})(x^{p^{i_2}} + y^{p^{i_2}})\cdots.$$

Hence, in the expansion, the resulting powers of x and y will only be raised to numbers which occur as sums of these p^{i_j}. One sees, then, that because $(6,3,3) = (3^1+3^1, 3^1, 3^1)$ and $(9,2,1) = (3^2, 3^0+3^0, 3^0)$ can be written as sums of different collections of powers of 3, the summands in their images under δ^2 *cannot interact*. This means that because $\zeta_3^{12} = (6,3,3) + (9,2,1) - (10,1,1)$ vanishes under δ^2, the polynomials $(6,3,3)$ and $(9,2,1) - (10,1,1)$ must both individually vanish as well.

It's still not clear how we came up with $(9,2,1)$, $(10,1,1)$, and $(6,3,3)$ to begin with — why are these the things that appear as summands in ζ_3^{12}? This is answered by a result of Kümmer: the multinomial coefficient $\binom{|\lambda|}{\lambda}$ has p-divisibility exactly equal to the number of times you perform a carry when summing the components of λ in p-adic arithmetic. Because we divided out by the greatest common factor in our definition of ζ_k^n, the remaining summands in its projection to $\mathbb{F}_p$ will be those whose associated exponent index has minimal p-adic carry count.

This minimality finishes the job for us. When we check the 2-cocycle condition for a given polynomial f, only two summands can contribute completely mixed terms: $f(x_0 + x_1, x_2, \ldots, x_n)$ and $f(x_0, x_1 + x_2, x_3, \ldots, x_n)$. Correspondingly, if checking the 2-cocycle condition for some symmetrized monomial (λ) results in an errant monomial $x_0^{\lambda_1'} x_1^{\lambda_1''} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ for $\lambda_1' + \lambda_1'' = \lambda_1$, the only other symmetrized monomial that has a hope of canceling this stray guy is $(\lambda_1', \lambda_1'' + \lambda_2, \lambda_3, \ldots, \lambda_n)$. Now, supposing that we start with a non-carry-minimal partition, we can iteratively use this property to bring together the power-of-p pieces that cause the carry and sum them up — thereby decreasing the carry of the partition. However, our key observation shows that we can't go the other way around. For example, consider the partition $(10,2)$ in characteristic 3. Applying the 2-cocycle condition to $(10,2)$ yields the sum $2x_0^{10}x_1x_2 + 2x_0^9x_1x_2^2 + 2x_0x_1^9x_2^2$, and the only hope of deleting the summand $2x_0^9x_1x_2^2$ is to try to cancel it off with $(9,3)$ — but once the 2 and the 1 in the exponent are joined into a 3, they can't be pulled back apart, and $(9,3)$ actually cannot contribute the required cancellation. So, only carry-minimal partitions can participate in a 2-cocycle, and a similar argument shows that the cocycles we've described so far cannot be broken down any further.

To summarize, we have a big theorem: Select a power-of-p partition λ of n with length k. Let $T^m\lambda$ denote the set of all possible partitions of the form $G_{i_1 j_1} \cdots G_{i_m j_m}\lambda$. Then, if either $m \leq p-2$ or if λ is the shortest power-of-p partition of n, then polynomial $\sum_{\mu \in T^m\lambda} c_\mu(\mu)$ is a cocycle, where c_μ is the coefficient of (μ) in ζ_{k-m}^n. In addition, cocycles formed in this manner give a basis for the space of modular cocycles. Here's an illustration of this procedure applied to $n = 12$, $2 \leq k \leq 6$:

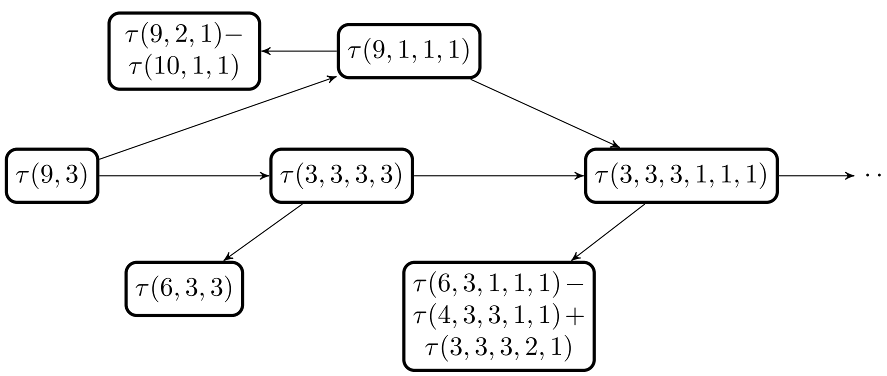

FIGURE 3. The homogeneous degree 12 part of $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$ for $2 \leq k \leq 6$.

4. 5/2

4.1. Rational multiplicative calculations. Once again, the rational calculation is effortless. We've asserted that the only additive cocycles available are those of the form ζ_k^n, and rationally we have an exponential function, and so the multiplicative 2-cocycle $\exp(\zeta_k^n)$ will do just fine. This solves the problem; you see immediately that

$$C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m) \times \operatorname{Spec} \mathbb{Q} = \operatorname{Spec} \mathbb{Q}[z_{n,k} \mid n \geq k].$$

4.2. The cohomology groups $H^*(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a)$. In the modular case, we don't have a way of exponentiating our tangent data into global data, and so we have to work a bit harder. To begin, there is a spectral sequence that takes local data as input and produces global data as output. Namely, filter the cochain complex computing $H^*(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$ by leading degree. The filtration quotients correspond to the homogeneous graded pieces of $H^*(\hat{\mathbb{G}}_a;\hat{\mathbb{G}}_a)$, also graded by total degree. This isn't exactly the source or the target we wanted, but it's a start, and it's worth our time to analyze it.

The source can be computed as follows[10]: Let a_i represent $x^{p^i} = \zeta_1^{p^i}$ and b_i represent $p^{-1}((x+y)^{p^i} - x^{p^i} - y^{p^i}) = \zeta_2^{p^i}$. Then,

$$H^*(\hat{\mathbb{G}}_a;\hat{\mathbb{G}}_a)(\mathbb{Q}) \cong \Lambda[b],$$
$$H^*(\hat{\mathbb{G}}_a;\hat{\mathbb{G}}_a)(\mathbb{F}_2) \cong \bigotimes_i \mathbb{F}_2[a_i],$$
$$H^*(\hat{\mathbb{G}}_a;\hat{\mathbb{G}}_a)(\mathbb{F}_p) \cong \left(\bigotimes_i \Lambda[a_i]\right) \otimes \left(\bigotimes_i \mathbb{F}_p[b_i]\right).$$

This is an exercise in homological algebra and application of the Tate resolution. The key ideas are that this boils down to computing $\mathrm{Ext}_{R[x]}(R,R)$ in the category of $R[x]$-comodules, that the P.D. algebra $\Gamma[x]$ is the dual of $R[x]$, and that this Ext-calculation is isomorphic to $\mathrm{Ext}_{\Gamma[x]}(R,R)$ in the category of $\Gamma[x]$-modules. From here, we split into cases:

(1) $\mathrm{Ext}_{\Gamma_{\mathbb{Q}}[x]}(\mathbb{Q},\mathbb{Q})$: The chain complex

$$0 \longleftarrow \mathbb{Q}[a] \xleftarrow{a \leftarrow\!\shortmid 1} \mathbb{Q}[b] \longleftarrow 0$$

is a projective resolution of $\mathbb{Q}$ with $\mathbb{Q}[a] \to \mathbb{Q}$ given by $a \mapsto 0$, so we compute $\mathrm{Ext}_{\Gamma_{\mathbb{Q}}[x]}(\mathbb{Q},\mathbb{Q})$ to be $\Lambda[b]$ as promised.

(2) $\mathrm{Ext}_{\Gamma_{\mathbb{F}_2}[x]}(\mathbb{F}_2,\mathbb{F}_2)$: The algebra $\Gamma_{\mathbb{F}_2}[x]$ splits as the tensor product $\Gamma_{\mathbb{F}_2}[x] \cong \bigotimes_{i=0}^\infty \Lambda_{\mathbb{F}_2}[x^{[2^i]}]$, hence it suffices to compute $\mathrm{Ext}_{\Lambda_{\mathbb{F}_2}[y]}(\mathbb{F}_2,\mathbb{F}_2)$ and then tensor together those results. The differential graded algebra described by Tate which computes $\mathrm{Ext}_{\Lambda[y]}(\mathbb{F}_2,\mathbb{F}_2)$ is given by $R_* = \Gamma[a] \cong \bigotimes_{i=0}^\infty \Lambda[a^{[2^i]}]$ with differential $da^{[j]} = a^{[j-1]}y$. Therefore,

$$\mathrm{Ext}_{\Gamma[x]}(\mathbb{F}_2,\mathbb{F}_2) \cong \bigotimes_{i=0}^\infty \mathrm{Ext}_{\Lambda[x^{[2^i]}]}(\mathbb{F}_2,\mathbb{F}_2) \cong \bigotimes_{i=0}^\infty \mathrm{Hom}(\Gamma[a_i],\mathbb{F}_2) \cong \bigotimes_{i=0}^\infty \mathbb{F}_2[a_i^\vee].$$

(3) $H^*(\hat{\mathbb{G}}_a;\hat{\mathbb{G}}_a)(\mathbb{F}_p)$: Just as before, $\Gamma_{\mathbb{F}_p}[x]$ splits as a product of algebras $\Gamma_{\mathbb{F}_p}[x] \cong \bigotimes_{i=0}^\infty T[x^{[p^i]}]$, where $T[y]$ denotes the truncated polynomial algebra $T[y] = \mathbb{F}_p[y]/y^p$. Hence, we reduce to calculating $\mathrm{Ext}_{T[y]}(\mathbb{F}_p,\mathbb{F}_p)$. The Tate differential graded algebra is described by $\Lambda[a]\otimes\Gamma[b]$, where $da = y$ and $db^{[j]} = ab^{[j-1]}y^{p-1}$. Therefore,

$$\mathrm{Ext}_{\Gamma[x]}(\mathbb{F}_p,\mathbb{F}_p) \cong \bigotimes_{i=0}^\infty \mathrm{Ext}_{T[x^{[p^i]}]}(\mathbb{F}_p,\mathbb{F}_p) \cong \bigotimes_{i=0}^\infty \mathrm{Hom}(\Lambda[a_i]\otimes\Gamma[b_i],\mathbb{F}_p) \cong \bigotimes_{i=0}^\infty \Lambda[a_i^\vee]\otimes\mathbb{F}_p[b_i^\vee].$$

5. 10/6

(We spent the majority of this day reviewing our previously accomplishments, since it had been a whole season since we'd last spoken about the subject.) Now, we turn to differentials in this spectral sequence. We're interested in the classes in $H^2(\hat{\mathbb{G}}_a;\hat{\mathbb{G}}_a)$. Over $\mathbb{F}_2$, such classes are of the form a_ia_j for distinct i,j, representable as $x^{2^i}y^{2^j}$. From this presentation, we compute that we get a nontrivial differential

$$d_{2^i+2^j}(ka_ia_j) = k^2a_i^2a_{j+1} - k^2a_{i+1}a_j^2.$$

Over $\mathbb{F}_p$, if we attempt the same thing we get a cohomologically trivial differential, which means we have to pick a smarter starting class. Namely, take ka_ia_j to be represented by $\mathrm{texp}(kx^iy^j)$. This gives the nontrivial differential

$$d_{p^i+p^j}(ka_ia_j) = k^p(a_{i+1}b_{j+1} - a_{j+1}b_{i+1}).$$

Together, these results say something important about extending additive cohomology classes to multiplicative ones: it's necessary that your leading coefficient k satisfy $k^p = 0$, or else you'll run into this obstruction.

[10] The computation was known at least as far back as Lubin and Tate in 1966, but the method presented here I've heard attributed to Hopkins.

It's worth emphasizing that this doesn't obstruct the classes we're interested in: our classes $\zeta_2^{p^i+p^j}$ are cohomologically of the form $a_i a_j + a_j a_i$, which is cohomologically null. So, a naïve computational goal then is to produce asymmetric classes related to the classes we're interested in, to which we can then apply this result.

Commentary: I also mentioned the end goal of the sequence of talks, as a hook for the audience. For an elliptic spectrum E, i.e. a spectrum E, elliptic curve C, and an isomorphism $\hat{C} \cong \mathbb{CP}^\infty_E$, we can use arithmetic geometry to produce a natural cubical structure $u \in C^k(\mathbb{CP}^\infty_E; \hat{\mathbb{G}}_m)(E_*)$. By our schematic analysis, this gives a multiplicative map $MU\langle 6\rangle_* \to E_*$, which for formal reasons lifts to a map of spectra $MU\langle 6\rangle \to E$. This assignment is natural in the choice of elliptic spectrum, and so gives a whole diagram of maps $MU\langle 6\rangle \to E$. Thinking of tmf as the homotopy limit in spectra of all available elliptic spectra (which is a dicey thing to do without justification), we therefore get a map $MU\langle 6\rangle \to tmf$. This is called "the σ-orientation," and on homotopy it realizes the Witten genus for complex manifolds. The *actual* Witten genus uses real *String*-manifolds, which is a step removed from what we'll accomplish.

To be continued!