Higher Coh. Ops.

☆ Everything here is <u>stable</u>. (Usually, this will mean $q \leq 2n-2$.)

Functional ops

Def. Let $f: A \to X$ (may as well be an inclusion); write $K = X \cup_f CA$. Let $\theta \in \mathcal{O}(G,n;\pi,q)$. Then θ gives a map of lexseq's.

Suppose $u \in H^*(X;G)$ s.t. $f^*u = 0$ and $\theta u = 0$.

$$\begin{array}{ccccccc}
 & & H^n(K;G) & & & & \\
 & & \| & & & & \\
H^{n-1}(A;G) & \xrightarrow{\delta} & H^n(X,A;G) & \longrightarrow & H^n(X;G) & \xrightarrow{f^*} & H^n(A;G) \\
\downarrow \theta & & \downarrow \theta & & \downarrow \theta & & \\
H^{q-1}(X;\pi) \xrightarrow{f^*} H^{q-1}(A;\pi) & \xrightarrow{\delta} & H^q(X,A;\pi) & \longrightarrow & H^q(X;\pi) & & \\
 & & \| & & & & \\
 & & H^q(K;\pi) & & & &
\end{array}$$

indeterminacy

$u' \longmapsto u \longmapsto 0$

$u'' \longmapsto \theta u'$

$$\theta_f : \left(\ker f^* \cap \ker \theta\right) \longrightarrow H^{q-1}(A;\pi)/Q$$

$\subseteq H^n(X;G)$ $\qquad Q = f^*(H^{q-1}(X;\pi)) + \theta(H^{n-1}(A;G))$

$u \longmapsto u''$

We will try to ignore indeterminacy for the sake of exposition, but in the real world it's often a very big deal.

naturality:

$$\begin{array}{ccc}
A & \xrightarrow{f} & X \\
g\uparrow & = & \uparrow h \\
A' & \xrightarrow{f'} & X'
\end{array}$$

If $\theta_f u$ exists, then $g^*(\theta_f u)$ "=" $\theta_{f'}(h^*u)$.

Fact: If f is nullhomotopic, then $\theta_f = 0$.

Example: Let $f: S^{n+1} \to S^n$ be a suspension of the Hopf map $\eta: S^3 \to S^2$. Take $\theta = Sq^2$ (so $G = \pi = \mathbb{Z}_2$).

$$\begin{tikzcd}
 & \overset{0}{H^{n-1}(S^{n+1})} \arrow[r] \arrow[d, "Sq^2"] & \overset{\mathbb{Z}_2}{H^n(K)} \arrow[r, "\sim"] \arrow[d, "Sq^2"] & \overset{\mathbb{Z}_2}{H^n(S^n)} \arrow[r] \arrow[d, "Sq^2"] & \overset{0}{H^n(S^{n+1})} \\
\underset{0}{H^{n+1}(S^n)} \arrow[r] & \underset{\mathbb{Z}_2}{H^{n+1}(S^{n+1})} \arrow[r, "\sim"] & \underset{\mathbb{Z}_2}{H^{n+2}(K)} \arrow[r] & \underset{0}{H^{n+2}(S^n)} &
\end{tikzcd}$$

$u' \mapsto u$

$v \mapsto Sq^2 u \neq 0$

(generator)

By definition of η, Sq^2 is nonzero on $H^2 \to H^4$ for $S^2 \cup_\eta e^4$. (Since this map has Hopf invariant 1, then in integral cohomology, with generators $x \in H^2(S^2 \cup_\eta e^4; \mathbb{Z})$, $y \in H^4(S^2 \cup_\eta e^4; \mathbb{Z})$, $Sq^2 x = x^2 = 1 \cdot y$.) Then since Sq^i are stable and $K = S^{n-2}(S^2 \cup_\eta e^4)$, the result follows.

We say that Sq^2 detects f.

reformulation:

$$\begin{tikzcd}
 & F = K(\pi, q-1) \arrow[r] & E \arrow[d] & \\
Y \arrow[r, "f"] \arrow[ur, dashed, "u''"] \arrow[rr, bend right, "uf \simeq 0"'] & X \arrow[r, "u"] \arrow[ur, dashed, "\tilde{u}"] \arrow[rr, bend right, "\theta u \simeq 0"'] & B = K(G,n) \arrow[r, "\theta"] & K(\pi, q)
\end{tikzcd}$$

$\boxed{\text{Secondary coh. ops.}}$

Def. Given 2-stage P. tower:

$$\begin{tikzcd}
 & F = K(\pi, q-1) \arrow[r, "i"] & E \arrow[r, "\varphi"] \arrow[d, "p"] & K(H, m) \\
 & X \arrow[r, "u"] \arrow[ur, dashed, "\tilde{u}"] \arrow[rr, bend right, "\theta u \simeq 0"'] & B = K(G,n) \arrow[r, "\theta"] & K(\pi, q)
\end{tikzcd}$$

$$\Phi: \underset{H^n(X;G)}{\operatorname{Ker} \theta_\#} \longrightarrow H^m(X; H)/Q, \quad u \mapsto \tilde{u}^* \varphi, \qquad Q = (\varphi i)_\#(H^{q-1}(X; \pi))$$

justification for name: Suppose $\varphi i \simeq 0$. Then $i^*(\varphi) = 0 \in H^m(F;H)$. So in Serre's lexseq., $\varphi \in \ker(i^*) = \operatorname{im}(p^*)$. So we have

$$\begin{tikzcd} & E \arrow[r, "\varphi"] \arrow[d, "p"] & K(H,m) \\ X \arrow[ur, "\tilde{u}"] \arrow[r, "u"'] & B \arrow[ur, dashed, "\theta'"'] & \end{tikzcd}$$

so $\Phi(u) = \theta'(u)$ (with no ambiguity).

naturality: up to indeterminacy, if $f: A \to X$, $u \in H^n(X;G)$, then if $\Phi(u)$ exists,

$$f^*(\Phi(u)) \text{"="} \Phi(f^* u).$$

relationship with relations: (by stability) We can uniquely write $\varphi i : K(\pi, q-1) \to K(H,m)$ as a desuspension ${}^1\psi$. Thus we extend the diagram

$$\begin{tikzcd} F = K(\pi,q-1) \arrow[r, "i"] \arrow[rr, bend left, "{}^1\psi"] & E \arrow[r, "\varphi"] \arrow[d, "p"] & K(H,m) & \\ & B = K(G,n) \arrow[r, "\theta"] & K(\pi,q) \arrow[r, "\psi"] & K(H,m+1). \end{tikzcd}$$

This ends up giving $\psi \circ \theta = 0 \in \mathcal{O}(G,n;H,m+1)$. Conversely, such a relation gives a secondary operation.

example: By Adem relation, $Sq^3 Sq^1 + Sq^2 Sq^2 = 0$.

$$\begin{tikzcd} F = K(\mathbb{Z}_2,n) \times K(\mathbb{Z}_2,n+1) \arrow[r] \arrow[rr, bend left, "Sq^3 \otimes id + id \otimes Sq^2"] & E \arrow[r] \arrow[d] & K(\mathbb{Z}_2, n+3) \\ & K(\mathbb{Z}_2,n) \arrow[r, "{(Sq^1, Sq^2)}"] & K(\mathbb{Z}_2,n+1) \times K(\mathbb{Z}_2,n+2) \end{tikzcd}$$

So if $u \in H^n(X)$ has $Sq^1 u = Sq^2 u = 0$, then we get $\Phi(u) \in H^{n+3}(X)/Q$.

(first) Peterson-Stein formula: This relates functional & secondary ops:

$$\begin{tikzcd} & F = K(\pi, q-1) \arrow[r, "i"] \arrow[rr, bend left, "'\psi"] & E \arrow[r, "\varphi"] \arrow[d, "p"] & K(H,m) & \\ Y \arrow[r, "f"] \arrow[ur, "\theta_f(u)"] & X \arrow[r, "u"] & B = K(G,n) \arrow[r, "\theta"] & K(\pi, q) \arrow[r, "\psi"] & K(H, m+1) \end{tikzcd}$$

$$f^*(\Phi(u)) = {}'\psi(\theta_f(u))$$

application

(also: Sq^1 detects $2 \in \pi_0^S$; more generally, the Bockstein d_r detects $2^r \cdot 2$)

Q: Let $\eta: S^{n+1} \to S^n$ be the suspension of $\eta: S^3 \to S^2$.

Then $\eta \in \pi_1^S$. What is $\eta \circ \eta = \eta^2 \in \pi_2^S$?

$$S^{n+2} \xrightarrow{\eta} S^{n+1} \xrightarrow{\eta} S^n$$

Note $\pi_2^S \cong \mathbb{Z}_2$, so we only need to see if $\eta^2 \simeq 0$.

Let: $K = S^{n+1} \cup_\eta e^{n+3}$

$L = S^n \cup_{\eta^2} e^{n+3}$

$f: K \to L$, $f = \begin{cases} id & \text{on } e^{n+3} \\ \eta & \text{on } S^{n+1} \end{cases}$

We have

$$\begin{tikzcd} K \arrow[r, "f"] & L \\ S^{n+1} \arrow[u, "\substack{\text{induces} \\ H^{n+1}\text{-iso.}}"] \arrow[r, "\eta"] & S^n \arrow[u, "\substack{\text{induces} \\ H^n\text{-iso.}}"'] \end{tikzcd}$$

As we calculated earlier, $Sq^2_\eta(\underset{H^n(S^n)}{\sigma^n}) = \underset{H^{n+1}(S^{n+1})}{\sigma^{n+1}}$. Since these are coh. iso's in the relevant dimensions, $Sq^2_f(\underset{H^n(L)}{\sigma^n}) = \underset{H^{n+1}(K)}{\sigma^{n+1}}$.

Define a secondary coh. op. Φ from the P. tower of S^n:

$$\begin{tikzcd}
 & F = K(\mathbb{Z}_2, n+1) \arrow[r, "i"] \arrow[rr, bend left, "\varphi i = {}^{1}\psi = Sq^2"] & X_{n+1} \arrow[r, "\varphi"] \arrow[d] & K(\mathbb{Z}_2, n+3) & \\
K \arrow[r, "f"] \arrow[ur, "Sq^2_f(\sigma^n) = \sigma^{n+1}"] \arrow[rr, bend right] \arrow[rrr, bend right, "\text{both} \simeq 0 \text{ for dimension reasons}"'] & L \arrow[r, "\sigma^n"] & B = K(\mathbb{Z}, n) \arrow[r, "\theta = Sq^2"] & K(\mathbb{Z}_2, n+2) \arrow[r, "\psi = Sq^2"] & K(\mathbb{Z}_2, n+4)
\end{tikzcd}$$

Now by the P-S formula,

$$f^*(\Phi(\sigma^n)) = Sq^2(Sq^2_f(\sigma^n)) = Sq^2(\sigma^{n+1}) \underset{\uparrow}{=} e^{n+3}.$$

Showed in previous example ("Sq^2 detects η")

But $f^* : H^3(L) \xrightarrow{\sim} H^3(K)$

so $\Phi(\sigma^n) = e^{n+3} \in H^3(L)$.

This is nonzero, so the attaching map η^2 is essential. $\square$