

# Higher Coh. Ops.

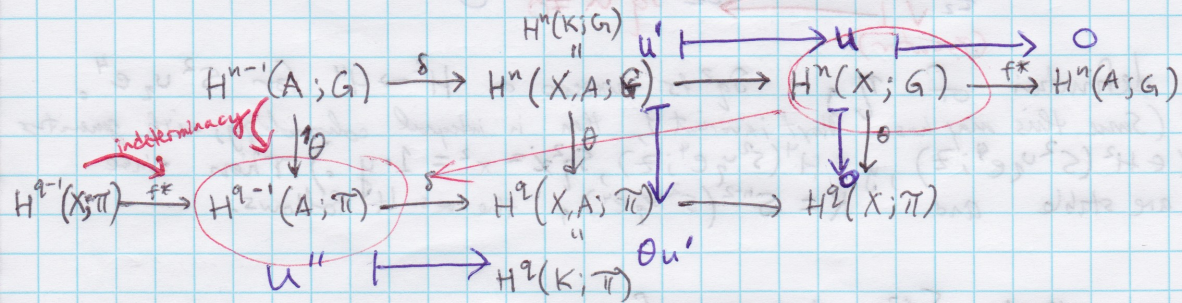
①

★ Everything here is stable. (Usually, this will mean  $q \leq 2n - 2$ .)

## Functional ops

Def. Let  $f: A \rightarrow X$  (may as well be an inclusion); write  $K = X \cup_f CA$ . Let  $\theta \in \mathcal{O}(G, n; \pi, q)$ . Then  $\theta$  gives a map of Serre's,

Suppose  $u \in H^*(X; G)$  s.t.  $f^*u = 0$  and  $\theta u = 0$ .



$$\theta_f : (\ker f^* \cap \ker \theta) \xrightarrow{\cong} H^{q-1}(A; \pi) / Q$$

$\begin{matrix} \xrightarrow{f^*} & H^n(X; G) & \xrightarrow{\theta} & H^q(X; \pi) \\ u & \xrightarrow{\quad} & u' & \end{matrix}$

$Q = f^*(H^{q-1}(X; \pi)) + \theta(H^n(A; G))$

We will try to ignore indeterminacy for the sake of exposition, but in the real world it's often a very big deal.

Naturality:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \uparrow g & & \uparrow h \\ A' & \xrightarrow{f'} & X' \end{array}$$

If  $\theta_{f'} u'$  exists, then  $g^*(\theta_f u) = \theta_{f'}(h^* u')$ .

Fact: If  $f$  is nullhomotopic, then  $\theta_f = 0$ .



Example: Let  $f: S^{n+1} \rightarrow S^n$  be a suspension of the Hopf map  $\eta: S^3 \rightarrow S^2$ . Take  $\Theta = Sq^2$  (so  $G = \mathbb{Z}_2$ ).

(2)

$$\begin{array}{ccccccc}
 \begin{array}{c} 0 \\ \parallel \\ H^{n+1}(S^{n+1}) \end{array} & \longrightarrow & \begin{array}{c} \mathbb{Z}_2 \\ \parallel \\ H^n(K) \end{array} & \xrightarrow{\sim} & \begin{array}{c} \mathbb{Z}_2 \\ \parallel \\ H^n(S^n) \end{array} & \longrightarrow & \begin{array}{c} 0 \\ \parallel \\ H^n(S^{n+1}) \end{array} \\
 \downarrow Sq^2 & & \downarrow Sq^2 & \xrightarrow{u'} & \downarrow Sq^2 & & \\
 \begin{array}{c} 0 \\ \parallel \\ H^{n+1}(S^n) \end{array} & \longrightarrow & \begin{array}{c} \mathbb{Z}_2 \\ \parallel \\ H^{n+1}(S^{n+1}) \end{array} & \xrightarrow{\sim} & \begin{array}{c} \mathbb{Z}_2 \\ \parallel \\ H^{n+2}(K) \end{array} & \longrightarrow & \begin{array}{c} 0 \\ \parallel \\ H^{n+2}(S^n) \end{array} \\
 \downarrow & & \downarrow & \xrightarrow{Sq^2 u} & \downarrow & & \\
 0 & & \mathbb{Z}_2 & \xrightarrow{Sq^2 u \neq 0} & 0 & & 0
 \end{array}$$

By definition of  $\eta$ ,  $Sq^2$  is nonzero on  $H^2 \rightarrow H^4$  for  $S^2 \cup_2 e^4$ . (Since this map has Hopf invariant 1, then in integral cohomology, with generators  $x \in H^2(S^2 \cup_2 e^4; \mathbb{Z})$ ,  $y \in H^4(S^2 \cup_2 e^4; \mathbb{Z})$ ,  $Sq^2 x = x^2 = 1 \cdot y$ .) Then since  $Sq^i$  are stable and  $K = S^{n+2}(S^2 \cup_2 e^4)$ , the result follows.

We say that  $Sq^2$  detects  $f$ .

reformulation

$$\begin{array}{ccccc}
 F = K(\pi, q-1) & \longrightarrow & E & \longrightarrow & K(\pi, q) \\
 \uparrow u'' & & \uparrow \tilde{u} & & \downarrow \\
 Y & \xrightarrow{f} & X & \xrightarrow{u} & B = K(G, n) \xrightarrow{\theta} K(\pi, q) \\
 \downarrow u f \approx 0 & & \downarrow \theta u \approx 0 & & 
 \end{array}$$

Secondary coh. ops.

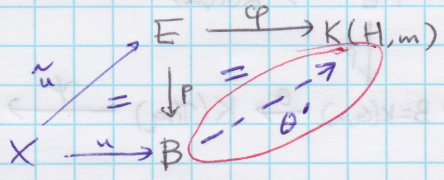
Given 2-stage P. tower:

$$\begin{array}{ccccc}
 F = K(\pi, q-1) & \xrightarrow{i} & E & \xrightarrow{\varphi} & K(H, m) \\
 \uparrow \tilde{u} & & \downarrow p & & \\
 X & \xrightarrow{u} & B = K(G, n) & \xrightarrow{\theta} & K(\pi, q) \\
 \downarrow \theta u \approx 0 & & & & 
 \end{array}$$

$$\Phi: \text{Ker } \theta \xrightarrow{u \mid \rightarrow \tilde{u}^* \varphi} H^m(X; H) / Q, \quad Q = (\varphi)_\# (H^m(X; \pi))$$



justification for name: Suppose  $\varphi_i \cong 0$ . Then  $i^*(\varphi) = 0 \in H^m(F; H)$ .  
 So in Serre's lexseq.,  $\varphi \in \ker(i^*) = \text{im}(p^*)$ . So we have

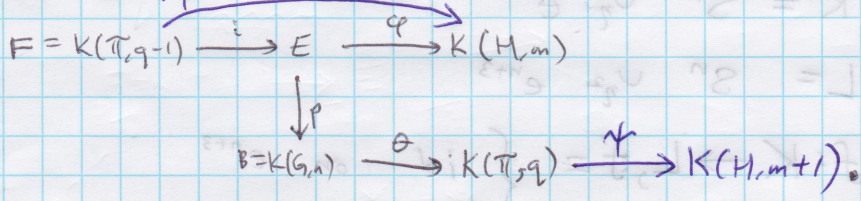


so  $\Phi(u) = \theta'(u)$  (with no ambiguity).

naturality: up to indeterminacy, if  $f: A \rightarrow X$ ,  $u \in H^*(X; G)$ , then if  $\Phi(u)$  exists,  $f^*(\Phi(u)) = \Phi(f^*u)$ .

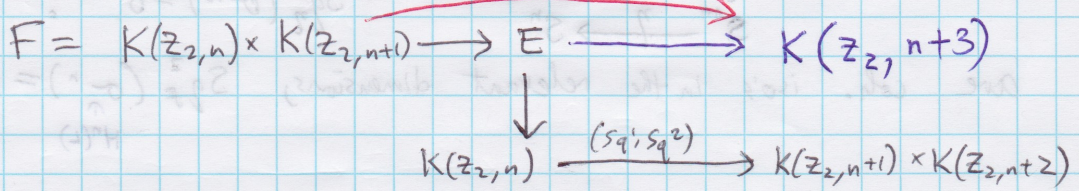
relationship with relations: We can uniquely write  $\varphi_i: K(\pi, q-1) \rightarrow K(H, m)$

as a desuspension  $\gamma_i$ . Thus we extend the diagram



This ends up giving  $\gamma_i \circ \theta = 0 \in \mathcal{O}(G, n; H, m+1)$ . Conversely, such a relation gives a secondary operation.

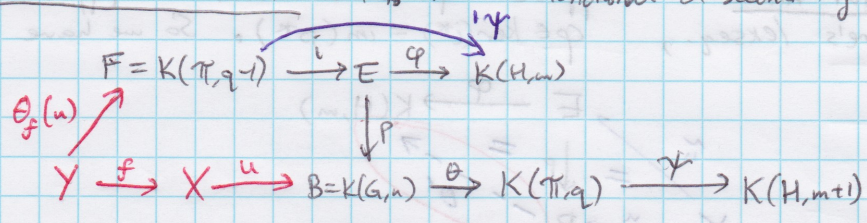
example: By Adem relation,  $Sq^3 Sq^1 + Sq^2 Sq^2 = 0$ .



So if  $u \in H^n(X)$  has  $Sq^1 u = Sq^2 u = 0$ , then we get  $\Phi(u) \in H^{n+3}(X)/Q$ .



(first) Peterson-Steen formula: This relates functional & secondary ops:



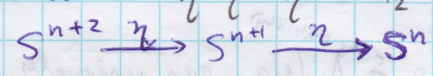
$$f^*(\Phi(u)) = \psi(\theta_f(u))$$

application

(also: Sq^0 detects  $z \in \pi_0^S$ ; more generally, the Bockstein  $d_r$  detects  $2^r \cdot z$ )

Q: Let  $\eta: S^{n+1} \rightarrow S^n$  be the suspension of  $\eta: S^3 \rightarrow S^2$

Then  $\eta \in \pi_{1,0}^S$ . What is  $\eta \circ \eta = \eta^2 \in \pi_2^S$ ?



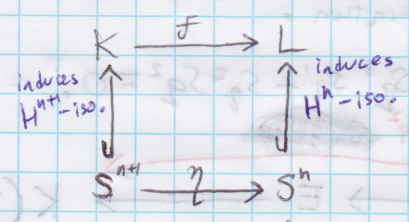
Note  $\pi_2^S \cong \mathbb{Z}_2$ , so we only need to see if  $\eta^2 \neq 0$ .

$$\text{Let: } K = S^{n+1} \cup_{\eta} e^{n+3}$$

$$L = S^n \cup_{\eta^2} e^{n+3}$$

$$f: K \rightarrow L, f = \begin{cases} \text{id} & \text{on } e^{n+3} \\ \eta & \text{on } S^{n+1} \end{cases}$$

We have



As we calculated earlier,

$$Sq_{\eta}^2(\sigma^n) = \sigma^{n+1}$$

are coh. iso's in the relevant dimensions,  $Sq_{f^*}^2(\sigma^n) = \sigma^{n+1}$



Define a secondary ch. op.  $\Phi$  from the P. tower of  $S^n$ : (5)

$$\begin{array}{ccccc}
 F = K(\mathbb{Z}_2, n+1) & \xrightarrow{i} & X_{n+1} & \xrightarrow{\varphi} & K(\mathbb{Z}_2, n+3) \\
 \uparrow \text{Sq}_f^2(\sigma^n) & & \downarrow & & \\
 K & \xrightarrow{f} & L & \xrightarrow{\sigma^n} & B = K(\mathbb{Z}_2, n) \\
 & & & \nearrow \text{both } \approx 0 \text{ for dimension reasons} & \\
 & & & & \xrightarrow{\theta = \text{Sq}_2^2} K(\mathbb{Z}_2, n+2) \xrightarrow{\eta = \text{Sq}_2^2} K(\mathbb{Z}_2, n+4)
 \end{array}$$

Now by the P-S formula,

$$f^*(\Phi(\sigma^n)) = \text{Sq}_2^2(\text{Sq}_f^2(\sigma^n)) = \text{Sq}_2^2(\sigma^{n+1}) \stackrel{\uparrow}{=} e^{n+3}.$$

But  $f^*: H^3(L) \xrightarrow{\sim} H^3(K)$   
 so  $\Phi(\sigma^n) = e^{n+3} \in H^3(L)$ .

shaded in previous example  
 ("Sq<sub>2</sub><sup>2</sup> detects η")

This is nonzero, so the attaching map  $\eta^2$  is essential.

