

# THE GEOMETRY OF FORMAL VARIETIES IN ALGEBRAIC TOPOLOGY I

ERIC PETERSON

ABSTRACT. Algebraic topology is full of computations with rings, and where we find rings we should seek geometry through methods of algebraic geometry. The geometry of formal varieties turn out to organize many interesting computations in topology, and certain formal varieties called commutative, one-dimensional formal groups give the best global picture of stable homotopy theory currently available. I will give as friendly an introduction to these ideas as can be managed; in particular, I will not assume the audience knows any algebraic geometry.

## 1. FORMAL VARIETIES

The impetus of algebraic geometry is to try to assign geometric meaning to concepts coming from ring theory. The classical example is that, given a set  $X$  of  $n$ -variate polynomials over an algebraically closed field  $k$ , we can associate the simultaneous vanishing locus of all elements of the set,  $Z(X)$ , in  $n$ -dimensional  $k$ -space.  $Z(X)$  is called an algebraic set, and irreducible (in a sense) algebraic sets are called varieties.

A key thing to notice here is that if you have a larger  $X$ , you have more polynomials, and so it gets harder for all of them to have zeroes in the same spots — and so the associated vanishing locus gets smaller. This means that, if we expect any kind of correspondence of this type between algebra and geometry, it has to be direction reversing.

A lot of energy was invested in figuring out what setting we can build that supports all rings and has nice geometric properties. Varieties, for instance, are insufficient; they turn out only to give a geometric description of finitely generated  $k$ -algebras with no nilpotent elements. Eventually, things called schemes were invented and people presented them as locally ringed spaces — this is what everyone sees in a first or second semester algebraic geometry course.

Since I come from topology, I find a different presentation of schemes much easier to think about: their functors of points. From a topological / categorical perspective, one presentation of the opposite category of rings is almost obvious: to every ring  $R$  we can build an object  $\text{Hom}(R, -)$ , which is a functor that assigns a ring  $S$  to the set of ring homomorphisms  $R \rightarrow S$ . The assignment  $\text{spec} : R \mapsto \text{Hom}(R, -)$  is contravariant (i.e., inclusion-reversing), and so provides an embedding of  $\text{Rings}^{op}$  into some larger category of functors off the category of rings, which I will call the category of schemes.

While not very obviously geometric, this idea of studying rings by studying the maps off them turns out to be a very good idea. In a sense, rather than asking “What is the ring  $R$ ?” it suggests that we instead ask “What does  $R$  do?” To illustrate what I mean, take the polynomial ring  $\mathbb{Z}[t]$  and associate to it the scheme  $\text{spec } \mathbb{Z}[t]$ . We can apply  $\text{spec } \mathbb{Z}[t]$  to a test ring  $S$  and see what we get:  $(\text{spec } \mathbb{Z}[t])(S) = \text{Hom}(\mathbb{Z}[t], S)$ . A map  $\mathbb{Z}[t] \rightarrow S$  is determined completely by where  $t$  is sent and  $t$  can be sent to any element — so  $\text{spec } \mathbb{Z}[t]$  models the forgetful functor  $\text{Rings} \rightarrow \text{Sets}$ . This is neat!

To get a sense of what else schemes can do, recall that each ring  $R$  comes with an additive group  $(R, +)$  on its underlying set of points, and maps between rings give rise to maps between these additive groups. The Yoneda lemma is a fundamental result of category theory, which in one language says that structure visible on the image of a representable functor actually lives on the representing object — and so we should expect a group structure on the scheme  $\text{spec } \mathbb{Z}[t]$ . Or, since  $\text{spec}$  is direction-reversing, we could call this a cogroup (or “Hopf algebra”) structure on the ring  $\mathbb{Z}[t]$ . To illustrate, we should get a map  $\mathbb{Z}[t] \rightarrow \mathbb{Z}[x] \otimes \mathbb{Z}[y]$  corresponding to the “opposite” of addition. In terms of the action of  $\text{spec}$ , to any two maps  $\mathbb{Z}[x] \rightarrow S$  and  $\mathbb{Z}[y] \rightarrow S$  (called  $S$ -valued points of the schemes  $\text{spec } \mathbb{Z}[x]$  and  $\text{spec } \mathbb{Z}[y]$ ), we need to associate a map  $\text{spec } \mathbb{Z}[t] \rightarrow S$  corresponding to their sum. The content of Yoneda’s lemma is that it’s enough to do this “in the universal case”: the identity maps  $\mathbb{Z}[x] \rightarrow \mathbb{Z}[x]$  and  $\mathbb{Z}[y] \rightarrow \mathbb{Z}[y]$  classify the elements  $x$  and  $y$ , and hence  $\mathbb{Z}[t] \rightarrow \mathbb{Z}[x] \otimes \mathbb{Z}[y]$  should classify the element  $x + y$  — and this is the opposite addition map we wanted.

This gives rise to a comultiplication map of  $\mathbb{Z}[t]$ , and so a multiplication map on  $\text{spec } \mathbb{Z}[t]$ .  $\text{spec } \mathbb{Z}[t]$  together with this group structure we call the “additive group (scheme),” written  $\mathbb{G}_a$ .

This happens *all the time* with all kinds of interesting functors on the category of rings. Another functor everyone knows is the group of units: there’s a functor  $\mathbb{G}_m$  with  $\mathbb{G}_m(R)$  equal to the invertible points of  $R$ . To get an  $R$ -point of  $\mathbb{G}_m$ , we need to select an element  $x$  in  $R$  and another element  $y$  playing the role of its inverse, so  $xy = 1$ . Using this logic, we can model  $\mathbb{G}_m$  as  $\text{spec } \mathbb{Z}[x, y]/\langle xy - 1 \rangle$ . This scheme is called the “multiplicative group.”

Shifting gears slightly, one of the classical presentations of infinitesimal elements are elements that “square to zero.” This turns out to be a fairly difficult thing to make both precise and analytically useful, which is why most analysts spend their time thinking about other things — but it’s a very useful notion in algebraic geometry. We think of the nilpotent elements of a ring  $R$  as being infinitesimal. Relatedly, there’s a very useful category similar to our category of schemes. Let  $R$  be a topologized augmented  $k$ -algebra whose augmentation ideal  $I$  is topologically nilpotent — the category of such objects is called the category of adic  $k$ -algebras (with continuous homomorphisms). We can do the same functor-of-points construction here, where the resulting schemes are called “formal schemes” and the version of  $\text{spec}$  is denoted  $\text{spf}$ . These are contravariant functors from adic  $k$ -algebras to **Sets**. (An element  $x$  is topologically nilpotent if for any  $n$  there’s an  $m$  with  $x^m \in I^n$ , where  $R$  has the  $I$ -adic topology. The elements “limit” to zero, so to speak.)

As an example, the ring of power series  $k[[t]]$  is an augmented adic  $k$ -algebra with the  $\langle t \rangle$ -adic topology, and so gives a formal scheme  $\text{spf } k[[t]] = \hat{\mathbb{A}}^1$ . This example, called the formal affine line, is particularly relevant — there’s a subcategory of formal schemes called formal varieties, consisting of those functors which are (noncanonically) isomorphic to  $\text{spf } k[[t_1, \dots, t_n]]$  for some  $n$ . The name “formal variety” comes from taking the sheaf of functions of a smooth algebraic variety and completing at a point. What you get is a  $k$ -algebra of this type, where the number of indeterminates is equal to the dimension of the variety. This is how you should think of a formal variety: just like a smooth manifold has about every point a neighborhood where it looks like Euclidean space, a smooth variety has about every point an infinitesimal neighborhood where it looks like “formal  $n$ -space,” and we’re interested in that tiny neighborhood.

Some more vocabulary: a formal variety  $V$  is something noncanonically isomorphic to  $\text{spf } k[[t_1, \dots, t_n]]$ . A selected isomorphism between  $V$  and  $\hat{\mathbb{A}}^n$  is called a “coordinate” on  $V$ , corresponding to picking charts on a manifold. The space of maps  $\hat{\mathbb{A}}^n \rightarrow \hat{\mathbb{A}}^m$  corresponds to  $m$ -tuples of  $n$ -variate power series, just like maps between charts on analytic manifolds correspond to power series.

## 2. VARIETIES ASSOCIATED TO COHOMOLOGY THEORIES

Let’s apply some of these words to algebraic topology. In first or second semester algebraic topology, one thing everyone learns about singular cohomology is that it supports a ring structure — there’s a notion of “cup product”  $H^n M \times H^m M \rightarrow H^{n+m} M$  for a space  $M$ , and so  $H^* M$  collectively forms a ring. As budding algebraic geometers, rather than trying to write down what these rings are, we know that we should try to write down what these rings “do.” This might not sit well with those of you that haven’t seen lots of algebraic topology; the examples given in an introductory course have very sparse information, but basically all spaces of lasting interest to a topologist have extremely complicated cohomological information embedded. The cohomology ring associated to  $S^n$ , for instance, isn’t so gripping — but  $K(\mathbb{Z}/p, q)$ , for instance, is.

At any rate, this suggests that rather than thinking of  $H^* M$  as a cohomology ring, we should think of  $M_H = \text{spf } H^* M$  as a formal scheme<sup>1</sup> over<sup>2</sup> the formal scheme  $S_H$ , which we define to be  $\text{spf } H^* \text{pt}$ .<sup>3,4</sup>

<sup>1</sup>The ideal inducing the  $I$ -adic topology corresponds to cohomological information above degree 0, using the functoriality of  $H^*$  and the maps  $\text{pt} \rightarrow M \rightarrow \text{pt}$ .

<sup>2</sup>“Over” here is used in the sense of a comma category, so if  $\text{spec } S$  is a scheme over  $\text{spec } R$ , that means we’ve included a map  $\text{spec } S \rightarrow \text{spec } R$  as part of the data associated to  $\text{spec } S$ . Since all such maps come from maps of rings, that means there’s a ring homomorphism  $R \rightarrow S$  inducing this map, and so  $S$  is an  $R$ -algebra. This is important information to specify, since just like you need a ground ring to form tensor products in commutative algebra, you need a base scheme to form fiber products in algebraic geometry.

<sup>3</sup>In fact,  $H$  can be replaced by any ring-valued cohomology theory to produce similar results. The  $H$  in  $M_H$  refers to singular cohomology, but, for example,  $M_K = \text{spf } K^0 M$  for complex  $K$ -theory will also be relevant.

<sup>4</sup>One thing the reader should note — and object to! — is that this construction is extremely insensitive to issues with grading. Namely, we’ve thrown away the natural grading given to us by cohomological dimension, and we’ve even assumed

So, what are some examples where this ideology is a good idea in topology? One calculation that everyone should know is the singular cohomology of infinite dimensional complex projective space:  $H^*\mathbb{C}P^\infty = \mathbb{Z}[[t]]$ , with  $t$  a cohomology class in degree 2.<sup>5</sup> So,  $\mathbb{C}P_H^\infty = \text{spf } H^*\mathbb{C}P^\infty$  is isomorphic to the formal affine line,  $\hat{\mathbb{A}}^1 = \text{spf } \mathbb{Z}[[t]]$ , where our choice of  $t$  in the cohomology of  $\mathbb{C}P^\infty$  gave us a coordinate on  $\mathbb{C}P_H^\infty$ . That's neat, albeit not too interesting; it's the most basic formal scheme we know of, pretty much.

To help make it interesting, there is a group structure on  $\mathbb{C}P^\infty$ , as the classifying space for line bundles, corresponding to the universal tensor product of line bundles. Let's study what structure this induces on the formal scheme associated to  $\mathbb{C}P^\infty$  by computing the product map  $\mathbb{C}P_H^\infty \times \mathbb{C}P_H^\infty \rightarrow \mathbb{C}P_H^\infty$  using the coordinate we have. We effectively need to ask what  $t$  pulls back to under the multiplication map  $H^*\mathbb{C}P^\infty \rightarrow H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ . Well, a priori, we have that it pulls back to some bivariate power series

$$F(x, y) \in H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong H^*\mathbb{C}P^\infty \hat{\otimes} H^*\mathbb{C}P^\infty \cong \mathbb{Z}[[x, y]].$$

The niceness of the tensor product tells us a few things about  $F$ : the tensor product is commutative, so  $F(x, y) = F(y, x)$ . The tensor product satisfies  $L \otimes 1 \cong L$  for any line bundle  $L$ , so  $F(x, 0) = x$ .<sup>6</sup> One thing we can deduce from this is that  $F(x, y) = x + y +$  (terms mixed in  $x$  and  $y$ ). Lastly, the tensor product is associative,  $(L \otimes M) \otimes N \cong L \otimes (M \otimes N)$ , so we have  $F(F(x, y), z) = F(x, F(y, z))$ . This is quite a bit of structure! Such an  $F$  is called a (commutative, one-dimensional) formal group law.

In ordinary cohomology, we can actually calculate  $F$ . The Künneth formula says that we have  $H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \mathbb{Z}[[x, y]]$  with  $x$  and  $y$  both of degree 2, so the only possible thing  $F$  could be is  $F(x, y) = x + y$  due to degree restrictions. — But this is already interesting! Another interpretation of this calculation is that the induced map  $\mathbb{C}P_H^\infty \times \mathbb{C}P_H^\infty \rightarrow \mathbb{C}P_H^\infty$  corresponding to the product on formal schemes is given by the power series  $F(x, y) = x + y$ , which, as we calculated earlier, is exactly the function corresponding to addition in the additive group. Hence,  $\mathbb{C}P_H^\infty$  is another name for  $\hat{\mathbb{G}}_a$ .

Let's do the same calculation for  $K$ -theory, another kind of cohomology theory, where  $K(X)$  corresponds basically to vector bundles over  $X$ . For some topological reasons<sup>7</sup>,  $K(\mathbb{C}P^\infty)$  also turns out to be  $K_*[[t]]$ , and so  $\mathbb{C}P_K^\infty$  is also an affine line over  $S_K$ .  $\mathbb{C}P_K^\infty$  comes with a natural choice of coordinate: namely, to the universal line bundle  $\mathcal{L}$  over  $\mathbb{C}P^\infty$ , we associate the  $K$ -theoretic class  $[\mathcal{L}] - 1$  corresponding to  $\mathcal{L}$ .<sup>8</sup> That tensor product corresponds to multiplication in  $K$ -theory means that we can calculate the action of the tensor product manually: write  $c_1(V)$  for the image of  $t$  under the pullback along a map  $M \rightarrow \mathbb{C}P^\infty$  classifying  $V$ . Then, where the first line is the definition, we calculate:

$$\begin{aligned} F(x, y) &= c_1(\mathcal{L} \otimes \mathcal{L}') \\ &= [\mathcal{L}][\mathcal{L}'] - 1 \\ &= [\mathcal{L}][\mathcal{L}'] - [\mathcal{L}] - [\mathcal{L}'] + 1 + [\mathcal{L}] - 1 + [\mathcal{L}'] - 1 \\ &= ([\mathcal{L}] - 1)([\mathcal{L}'] - 1) + [\mathcal{L}] - 1 + [\mathcal{L}'] - 1 \\ &= c_1(\mathcal{L})c_1(\mathcal{L}') + c_1(\mathcal{L}) + c_1(\mathcal{L}') \\ &= xy + x + y. \end{aligned}$$

If you work it out, this is exactly the product structure associated to the formal multiplicative group  $\hat{\mathbb{G}}_m$ , and hence  $\mathbb{C}P_K^\infty$  is isomorphic to  $\hat{\mathbb{G}}_m$ !<sup>9</sup>

that the odd-dimensional parts vanish, so that we get a strictly commutative ring. This is a lot to ask, and introduces a few problems that we'll just ignore. :)

<sup>5</sup>Those especially picky might also object here to the use of power series rather than a polynomial ring; this is a matter of taste in taking a limit or a colimit when building the cohomology ring from its graded pieces. Since  $\mathbb{C}P^\infty$  is the colimit of the  $\mathbb{C}P^n$  and  $H^*$  is contravariant, I'll take  $H^*\mathbb{C}P^\infty$  to be the limit of the rings  $H^*\mathbb{C}P^n \cong \mathbb{Z}[t]/\langle t^n \rangle$ , i.e.,  $H^*\mathbb{C}P^\infty \cong \mathbb{Z}[[t]]$ .

<sup>6</sup>To explain, the trivial line bundle is classified by the constant map to  $\mathbb{C}P^\infty$ , which induces the zero map on reduced cohomology, so the map classifying 1 pulls the coordinate  $t$  back to 0.

<sup>7</sup>The coefficient ring of  $K$  is even-concentrated and  $H^*\mathbb{C}P^\infty$  is even concentrated, so the Atiyah-Hirzebruch spectral sequence computing  $K(\mathbb{C}P^\infty)$  collapses.

<sup>8</sup>The  $-1$  is to fix the problem that the trivial bundle 1 should correspond to the 0 element in the formal variety, as before. Another way to think of this is that  $[\mathcal{L}]$  corresponds to the total Chern class of  $\mathcal{L}$ , but our coordinate is just supposed to detect the first Chern class — so we use  $[\mathcal{L}] - 1$  instead.

<sup>9</sup>But what is  $\hat{\mathbb{G}}_m$ ? It's supposed to be a formal group, so in particular a formal variety, so in particular isomorphic to  $\hat{\mathbb{A}}^1$  — but this is different from the usual multiplicative group, which looks like the variety  $\mathbb{A}^1 \setminus \{0\}$ . The property we want to capture is

Briefly, let's touch on some other, less pivotal examples:

- $BU(n)_H$  is isomorphic to  $n$ -dimensional affine space, with group structure corresponding to divisors of degree  $n$  on  $S_H$ .
- In the limit,  $BU_H$  corresponds to the formal group of divisors on  $S_H$ .
- At a prime  $p$ , the even part of  $\text{spec } H_*K(\mathbb{Z}, 3)$  turns out to model the functor of Weil pairings  $\hat{\mathbb{G}}_a^2 \rightarrow \hat{\mathbb{G}}_m$ .
- If you're familiar with Morava  $K$ -theory,  $K(\mathbb{Z}, *)_{K(n)}$  is the free ring object on  $B\mathbb{Z}/p_{K(n)}^\infty$ , which itself is isomorphic to  $\mathbb{C}P_{K(n)}^\infty[p^\infty]$ . This is also tied in with Weil pairings on  $p$ -divisible groups.
- A connective cover of  $K$ -theory  $\text{spec } E_*BU\langle 6 \rangle$  models cubical structures on the formal group associated to  $E$ -theory, giving rise to the Ando-Hopkins-Strickland(-Witten-Ochanine-Breen-Serre-...?)  $\sigma$ -orientation of elliptic cohomology theories. This is true in some generality; the even part of  $\text{spec } E_*BU\langle 2k \rangle$  should represent certain classes of multiextensions of  $\mathbb{C}P_E^\infty$  by  $\hat{\mathbb{G}}_m$ .

I don't really want to explain any of these examples; I just want to point out that this geometric language organizes many other huge calculations elsewhere in topology too.

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We'll continue next week with complex oriented theories, Quillen's theorem, and the moduli stack of formal groups.

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the multiplication operation in our adic  $R$ -algebra  $S$ , so we make use of nilpotence: to each element  $s$  in the augmentation ideal of  $S$ , we have an element  $1+s \in S$ . Two such elements multiply together to get  $(1+s)(1+t) = 1+s+t+st = 1+(s+t+st)$ , so the map multiplication map  $\hat{\mathbb{G}}_m \times \hat{\mathbb{G}}_m \rightarrow \hat{\mathbb{G}}_m$  is  $(s, t) \mapsto s+t+st$ . The map  $s \mapsto 1+s$  translates the topologically nilpotent neighborhood of zero to 1, so the convergent power series  $(1+s)^{-1} = \sum_{i=0}^{\infty} (-1)^i s^i$  gives the inverse map as a map of formal varieties.