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Kinds of K-theory

connective K-theory

Algebraic K-theory ⇝ algebraic K-theory of spaces

$0 \to N \to E \to B \to 0$

$[E] = [N] + [B]$

Euler characteristic

Serre-Swan ⇝ $\mathcal{F}$ over $\mathcal{O}_X$ ⟷ $V \downarrow B$

Topological K-theory of spaces

$\mathbb{C}^n$-bundles over X

impetus ↓ ↑ $K(1)$

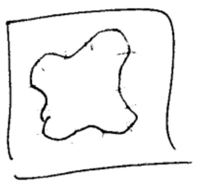
Morava K-theories

$E \simeq \bigvee_\alpha \Sigma^* K(n)$

$\forall$ $E \forall X$ E^*X is free

Def'n of a v.b.

"Continuous assignment of vector spaces to points."



- $E \xrightarrow{\pi} B$, U_α cover B, $\pi^{-1} U_\alpha \overset{g_\alpha}{\cong} U_\alpha \times \mathbb{C}^n$, and

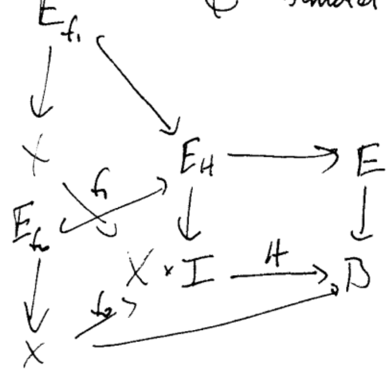
$$\mathbb{C}^n \xrightarrow[\text{at } p]{\text{include}} U_{\alpha\beta} \times \mathbb{C}^n \xrightarrow{g_\alpha^{-1}} \pi^{-1} U_{\alpha\beta} \xrightarrow{g_\beta} U_{\alpha\beta} \times \mathbb{C}^n \xrightarrow{\text{forget except } p} \mathbb{C}^n$$

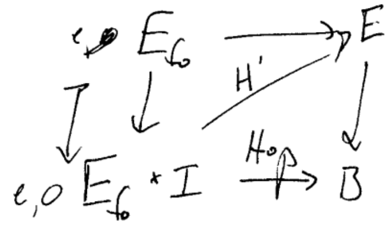
should be a linear isomorphism of vector spaces.

- Note that $U_n \circlearrowright \mathbb{C}^n$, and if we understand $U_n \circlearrowright E$ then we can recover E — a $U(n)$-bundle can be thickened to a v.b. by $U(n) \times_{U(n)} \mathbb{C}^n$ fiberwise.
- $\Pi_\infty B \longrightarrow$ Vector Spaces $\hookrightarrow$ Spaces.

∞-colimit ↓ ↑

$\mathbb{C}^n$-bundles over B.





The map $E_{f_0} \to E_{f_1}$ is defined by taking $H'|_{E_{f_0} \times \{1\}}$.

Maps $\Pi_\infty X \longrightarrow$ VectorSpaces should be "like" continuous maps $X \longrightarrow |\text{VectorSpaces}|$. What is something like $|\text{VectorSpaces}|$?

- Classifying spaces: to a topological group / H-group / category / $\cdots$ G we can associate a space BG so that $X \longrightarrow BG \leftrightsquigarrow G\text{-bundle}/X$.

  The identity $BG \xrightarrow{id} BG$ gives a G-bundle $G \to EG \to BG$, and EG is contractible, and uniquely determined by this property.

  pt, S^1

  $S^\infty \simeq \mathbb{C}^\infty_{\setminus\{0\}} \longrightarrow \mathbb{CP}^\infty$, for instance, is a model for $EU(1) \to BU(1)$.

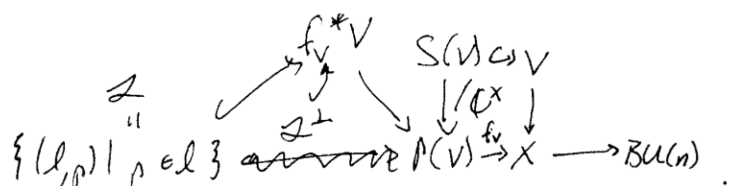
  Recall $\mathbb{CP}^\infty \simeq K(\mathbb{Z},2) \simeq BK(\mathbb{Z},1) \simeq BS^1 \simeq BU(1)$.

- Similarly, $\exists\, BU(n)$ which classify rank n vector bundles. Let's study $H^* BU(n)$ and $\pi_* \varinjlim BU(n) =: \pi_* BU$.

The Splitting principle:

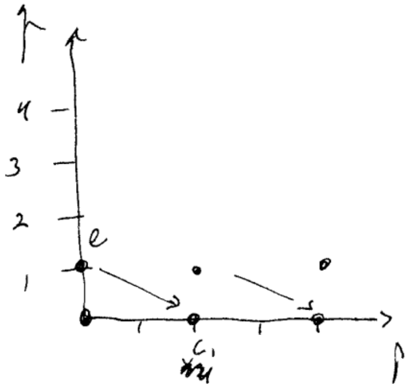
Maps $BU(n) \times BU(m) \xrightarrow{\oplus} BU(n+m)$
$BU(n) \times BU(m) \xrightarrow{\otimes} BU(nm)$

Every vector bundle $X \xrightarrow{V} BU(n)$ can be pulled back along $P(V) \xrightarrow{f_V} X$ to a sum of line bundles, and $H^*X \xhookrightarrow{H^* f_V} H^* P(V)$.



$H^* \mathbb{CP}^\infty = H^*(\mathbb{C}^0 \cup \mathbb{C}^1 \cup \mathbb{C}^2 \cup \cdots) = \{\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \cdots\}$, or

$H^p(\mathbb{CP}^\infty; H^q S^1) \Longrightarrow H^{p+q} EU(1) \simeq \mathbb{Z}$.



$\rightsquigarrow H^* \mathbb{CP}^\infty \cong \mathbb{Z}[c_1]$.

$H^*(\mathbb{CP}^\infty)^{\times k} = \mathbb{Z}[c_1, \ldots, c_k]$.

$$e \longmapsto (e, (0, \dots, 1))$$

$$U(n-1) \hookrightarrow U(n) \circlearrowleft EU(n) \longrightarrow EU(n) \times_{U(n)} S^{2n-1} \qquad BU(n)$$

$$\downarrow \qquad\qquad \downarrow$$

$$BU(n-1) \simeq EU(n)/U(n-1) \longrightarrow EU(n) \times_{U(n)} S^{2n-1} / U(n-1) \simeq$$

Maybe best to just say: Similar fiber sequence exist of the type

$$S^{2n-1} \longrightarrow BU(n-1) \longrightarrow BU(n),$$

and similar analysis with the SSS shows $H^*BU(n) \cong \mathbb{Z}[c_1, \dots, c_n]$.

And, in the end, we can take a colimit to get $H^*BU \cong \mathbb{Z}[c_1, c_2, \dots]$,

$\cong \mathrm{Sym}(H^*\mathbb{C}P^\infty)$.

$\Updownarrow$

Chern classes.

Let V/X be a bundle, and split to $L_1 \oplus \cdots \oplus L_n / Y$. Then we get a sequence of cohomology classes $c_1^V, \dots, c_n^V$ in H^*X determined by pulling back certain classes $c_1, \dots, c_n$ defined over $H^*(\mathbb{C}P^\infty)^{\times n}$ determined by $c_1\begin{pmatrix}\mathbb{C}^\infty \\ \downarrow \\ \mathbb{C}P^\infty\end{pmatrix} = c_1$, $c_0\begin{pmatrix}\mathbb{C}^\infty \\ \downarrow \\ \mathbb{C}P^\infty\end{pmatrix} = 1$, and $c(V \oplus W) = c(V) \cdot c(W)$.

A priori, these give cohomology classes in H^*Y by pullback, but these classes secretly live in $H^*BU(n)$.

The homotopy of BU.

Define $BU = \varinjlim BU(n)$, the classifying space for "stable" vector bundles. This guy has an inverse — he classifies stable equivalence classes of difference bundles.

$BGr_n \simeq BU$ (classifying space of frames) — space of self-maps of frames

$BBU \simeq B\left(\coprod_n BU(n)\right)$ (gp completion)

$$\left\{\frac{n \text{ dim'l subspaces}}{\text{identities}}\right\} \longleftarrow \left\{\begin{matrix}\text{inclusions } \mathbb{C}^n \hookrightarrow \mathbb{C}^\infty, \\ \text{unitary autos } \theta, f\end{matrix} \quad \begin{matrix}\mathbb{C}^\infty \xrightarrow{\theta} \mathbb{C}^\infty \\ \uparrow \quad \nearrow \\ \mathbb{C}^n\end{matrix}\right\} \longrightarrow \left\{\frac{n \in \mathbb{Z}}{\text{unitary autos of } \mathbb{C}^n}\right\}$$

Each of these are simplicial objects.

Snaith's theorem

So, $BBU \cong BU$, or $\Omega^2 BU \xrightarrow{\simeq} BU$, or $\exists\ S^2 \longrightarrow BU$ which we've found an inverse for — this is $[L]-1$, $L = S^3/S^2$.

$$\mathbb{C}P^\infty \xrightarrow{[L]-1} BU$$
$$\wr$$
$$\Sigma^\infty_+ \mathbb{C}P^\infty \longrightarrow KU$$
$$\wr$$
$$\Sigma^\infty_+ \mathbb{C}P^\infty[\beta^{-1}] \longrightarrow KU.$$

It turns out that this is a homotopy equivalence of spectra. Gepner has given a nice proof of this; I'd like to point out one part:

$M = R_\circ \mathbb{C}P^\infty$ $\quad R^\circ_\ast \mathbb{C}P^\infty$ ($\cong$) $\qquad R$ is an even-periodic ring spectrum.

$A = \operatorname{Sym} M$

$I = \operatorname{Aug} A$

$$0 \to [I/I^2, R] \xrightarrow{\cong} \operatorname{Add}(BU, \Omega^\infty R) \longrightarrow 0$$

$$0 \longrightarrow [I,R] \longrightarrow [A,R] \longrightarrow [R,R] \longrightarrow 0$$

$$0 \longrightarrow \tilde{R}^\circ BU \longrightarrow R^\circ BU \longrightarrow R^\circ pt \longrightarrow 0$$

$$0 \longrightarrow [I\otimes I, R] \longrightarrow [A\otimes A, R] \longrightarrow [R\oplus I\oplus I, R] \longrightarrow 0$$

$$0 \to \tilde{R}^\circ BU\wedge BU \longrightarrow R^\circ BU\times BU \longrightarrow R^\circ BU \vee BU \longrightarrow 0$$

This lifts to an equivalence $\operatorname{maps}(\Sigma^\infty_+ \mathbb{C}P^\infty, R) \longrightarrow \operatorname{maps}(KU, R)$

$$\rightsquigarrow \operatorname{Maps}(\Sigma^\infty_+ \mathbb{C}P^\infty \times \mathbb{Z}, R)\ ?$$