DETERMINANTAL *K*-THEORY AND A FEW APPLICATIONS

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ABSTRACT. Chromatic homotopy theory is an attempt to divide and conquer algebraic topology by studying a sequence of what we'd first assumed to be "easier" categories. These categories turn out to be very strangely behaved — and furthermore appear to be equipped with intriguing and exciting connections to number theory. To give an appreciation for the subject, I'll describe the most basic of these strange behaviors, then I'll describe an ongoing project which addresses a small part of the "chromatic splitting conjecture". This talk is meant to contain something for the novice and for the expert.

1. AN APPARENT CONNECTION TO ARITHMETIC

In algebraic topology, we are very concerned with understanding the homotopy groups of spheres. It's clear at least why algebraic topologists are interested in these groups: they encode structural data about the way cell complexes are assembled. They also seem to be *extremely* hard to compute — and so the sector we as a society understand correlates to our technological maturity in algebraic topology. No large collection of them has been computed without some corresponding major theoretical breakthrough.

To date, the most successful and encompassing program for organizing these groups goes under the banner of "chromatic homotopy theory." Chromatic homotopy theory makes central use of the homology theory BP, which is a slight modification of p-localized complex bordism. Its coefficient ring is polynomial:

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n, \ldots], \quad |v_n| = 2(p^n - 1).$$

It is motivated to a large extent by an intensive study of the BP-Adams spectral sequence, in which one can spot a great many patterns that are periodic against multiplication by powers of these generators v_n. This, in turn, pushes chromatic homotopy theorists to study the homology theories $v_n^{-1}BP$ as n varies, in an effort to understand these patterns one at a time.[1]

Each of these homology theories $v_n^{-1}BP$ comes with its own Adams spectral sequence, and Bousfield's theory of localization shows that each of these spectral sequences converges to the homotopy of some spectrum $L_n\mathbb{S}^0$. These L_n functors come equipped with natural transformations $L_n \to L_{n-1}$, begetting a tower

$$\cdots \to L_n X \to L_{n-1}X \to \cdots \to L_1 X \to L_0 X.$$

When X is a finite spectrum (in particular, when $X = \mathbb{S}^0$), there is an equivalence $X_{(p)} \simeq \lim_n L_n X$, called "chromatic convergence." This is a sort of sanity theorem, reassuring us that organizing the homotopy groups of spheres using this tower is a reasonable program.

Motivated by this last theorem, we can build a spectral sequence which encapsulates the chromatic stratification of the finite spectrum X. Define the *nth monochromatic layer of X* to be the fiber in the fiber sequence

$$M_n X \to L_n X \to L_{n-1} X.$$

There is then the "geometric chromatic spectral sequence" of signature

$$E^1_{s,n} = \pi_s M_n X \Longrightarrow (\pi_s X) \otimes \mathbb{Z}_{(p)}.$$

We thus arrive at the study of the groups $\pi_* M_n \mathbb{S}^0$ — and this is where the promised intriguing miracles begin to appear. First, there is the following theorem of Mark Behrens:

[1] We're also motivated by some remarkable formulas associated with BP-theory which link it to the theory of formal Lie groups. A comprehensive introduction would be entirely couched in that language, but that's not my goal here, so I will instead entirely avoid it.

Theorem (Mark Behrens). *Take $p \geq 5$. A certain element $\beta_{i/j,k}$, defined abstractly by properties, exists in the homotopy groups $\pi_* M_2 \mathbb{S}^0$ if and only if there exists a p-adic modular form $f_{i/j,k}$ of a particular weight depending upon i and satisfying particular congruences dependent upon j and k.*[2]

So, the homotopy of $M_2\mathbb{S}^0$ — while very difficult to express — encodes something about p-adic modular forms. The homotopy of $M_1\mathbb{S}^0$ is not so difficult to express:

Theorem (J. Frank Adams, as interpreted by Michael Hopkins). *Take $p \geq 3$. Let $\bar{\zeta}$ denote the composite*

$$\bar{\zeta} : \mathbb{N} \xrightarrow{z \mapsto \zeta(1-z)} \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}_{(p)} = \mathbb{Z}/p^\infty,$$

where ζ is the Riemann ζ-function. The homotopy $\pi_* M_1 \mathbb{S}^0$ takes the form

$$\pi_* M_1 \mathbb{S}^0 = \Sigma^{-2}\mathbb{Z}/p^\infty \oplus \bigoplus_{s=-1+|v_1|k} \Sigma^s \langle \bar{\zeta}|k| \rangle,$$

where $\langle \bar{\zeta}|k| \rangle$ denotes the subgroup of $\mathbb{Z}/p^\infty$ generated by the value $\bar{\zeta}|k|$.[3,4]

Suddenly, serious arithmetic has begun to appear in our study of algebraic topology! This means we are doing something of interest to more than just topologists — everyone is interested in arithmetic, to one degree or another.

Of course, if our intended goal was to understand the homotopy groups of spheres, we have not done a very good job of reducing the problem. Knowing that homotopy groups are expressable in terms of number theory won't make our lives any easier, as number theory is a difficult subject, nor will calculations in homotopy theory be much help in resolving quantitative questions in number theory, because this connection is fuzzily defined. However, the rich structure of these fields may help shed some light on the *qualitative* behavior of these groups, and the "chromatic splitting conjecture" is meant to predict the behavior of the divisible groups $\mathbb{Z}/p^\infty$ in the above formulas as n varies.

2. THE $K(n)$-LOCAL CATEGORY

Having now communicated some of the importance of studying monochromatic layers, I want to tell you a bit about what it feels like to actually do so. The right way to study any invariant, including monochromatic layers, is to find a "good" category in which it's valued and then study that category thoroughly. Such a category exists: it is the category of $K(n)$-local spectra.

Theorem. *There is a spectrum $K(n)$ with the property that $M_n L_{K(n)} \simeq M_n$ and $L_{K(n)} M_n \simeq L_{K(n)}$ (i.e., the homotopy types $L_{K(n)}X$ and $M_n X$ determine each other, functorially).*

The spectrum $K(n)$ has many reasonable properties that make it pleasant to study: most important of all, it is a "skew field spectrum", meaning that it has Künneth isomorphisms.[5] In the rest of this section, I will describe three essential facts about the $K(n)$-local category, but before we proceed I'd like to give a more proper statement of the chromatic splitting conjecture from a moment ago:

Conjecture (Michael Hopkins). *The nonzero torsion-free classes in $\pi_* L_{K(n)}\mathbb{S}^0$ belong to an exterior algebra on classes $x_1, x_3, \ldots, x_{2n+1}$ with $x_{2i+1} \in \pi_{-(2i+1)} L_{K(n)} \mathbb{S}^0$.*

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[2]For the curious, the specific conditions are: $f_{i/j,k}$ is of weight $i|v_2|/2$; $f_{i/j,k}(q) \not\equiv 0 \pmod p$; either $\mathrm{ord}_q f_{i/j,k}(q) > (i|v_2| - j|v_1|)/24$ or $\mathrm{ord}_q f_{i/j,k}(q) = (i|v_2| - j|v_1| - 2)/24$; there does not exist a modular form g of lower weight with $f_{i/j,k}(q) \equiv g(q) \pmod{p^k}$; and for every prime $\ell \neq p$ there exists a form $g_\ell \in M_{(i|v_2|-j|v_1|)/2}(\Gamma_0(\ell))$ satisfying $f_{i/j,k}(q^\ell) - f_{i/j,k}(q) \equiv g_\ell(q) \pmod{p^k}$.

[3]When $k = 0$, we encounter the singularity at $\zeta(1)$. At this point, I intend $\langle \bar{\zeta}(0) \rangle$ to mean the whole group $\mathbb{Z}/p^\infty$.

[4]Actually, this theorem is superficially misstated. The degree (-2) divisible factor is what really belongs to the pattern with the ζ-function, rather than the divisible factor in degree (-1). The maximum cuteness of this theorem is thwarted by discussing $\pi_* M_1 \mathbb{S}^0$, rather than $\pi_* L_{K(1)}\mathbb{S}^0$.

[5]In fact, Morava K-theories and ordinary homology with field coefficients form an exhaustive list of field spectra. So, even if you weren't interested in the chromatic program, you would run into these spectra as natural examples.

The full statement of this conjecture is a lot stronger and a lot more interesting — this is actually its least interesting part. It has been around for just under two decades now, with little progress made toward any part of it.[6]

2.1. **The $K(n)$-local Picard group.** Here's a less pretentious way to describe Adams's calculation:

$$\pi_s L_{K(1)}\mathbb{S}^0 = \begin{cases} \mathbb{Z}_p & \text{when } s=0, \\ \mathbb{Z}_p/(kp) & \text{when } s=-1+k|v_1|, \\ 0 & \text{otherwise.} \end{cases}$$

This formula suggests a way the connection to the ζ function could be strengthened: the quotient in the formula still makes sense when k is taken to be a general p-adic integer; any good notion of "p-adic ζ-function" should also be well-behaved on p-adic integer inputs; and these two situations could be compared. Iwasawa theory in number theory assures us that a p-adic ζ-function does exist, and it moreover agrees with the right–hand side of our formula. However, in terms of topology, this leads to an interesting question: if k is a p-adic integer, then s could be too — that is, we would be obligated to make sense of the mapping set $[\mathbb{S}^s, M_1\mathbb{S}^0]$, where s is a p-adic integer. What should this mean?

What makes a sphere, anyway? Here's one answer: in the classical category, the 0-sphere $\mathbb{S}^0$ is the unit of the smash product, and the s-sphere $\mathbb{S}^s$ has the property $\mathbb{S}^s \wedge \mathbb{S}^{-s} \simeq \mathbb{S}^0$. It turns out that the spheres are actually the only spectra with inverses, and hence one characterization is that spheres are the elements of the *Picard group* of the stable category. In the $K(n)$-local setting, when $n=1$ and $p \geq 3$, there is an encouraging calculation:

$$\operatorname{Pic}(L_{K(1)}\mathsf{Spectra}) \cong \mathbb{Z}_p \times \mathbb{Z}/|v_1|.$$

In particular, we have a factor of p-adic integers; we could hope that this factor gives our p-adic interpolation of the homotopy groups $\pi_* L_{K(1)}\mathbb{S}^0$ — and it indeed turns out to do so.[7]

2.2. **Gross–Hopkins duality.** Essentially all familiar geometric duality phenomena appearing in the study of homology (e.g., Poincaré, Alexander, and Atiyah dualities) can all be fit into the framework of Spanier–Whitehead duality, where the Spanier–Whitehead dual of a spectrum X is $DX = F(X, \mathbb{S}^0)$. The category of spectra also allows for the commingling of algebra with topology, and indeed there is a different sort of duality stemming from Pontryagin duality: the Brown–Comenetz dual of X is defined by the property

$$\pi_0 IX = \operatorname{Hom}(\pi_0 X, \mathbb{Q}/\mathbb{Z}).$$

This second construction is a rich source of counterexamples and generally strange behavior — to point out the dramatic difference between the two, we have an equivalence $D\mathbb{S}^0 \simeq \mathbb{S}^0$, whereas $I\mathbb{S}^0$ is coconnective.

In the $K(n)$-local category, however, the two constructions nearly coincide.

Theorem (Michael Hopkins and Benedict Gross). *Set $\hat{D}X = F(X, L_{K(n)}\mathbb{S}^0)$ and $\hat{I}X$ to be the spectrum with $\pi_0\hat{I}X = \operatorname{Hom}(\pi_0 M_n X, \mathbb{Q}/\mathbb{Z})$. In the $K(n)$-local category, there is a natural equivalence*

$$\hat{I}X \simeq \hat{D}X \wedge \hat{I}\mathbb{S}^0,$$

and $\hat{I}\mathbb{S}^0$ is a $\wedge$-invertible spectrum (i.e., a generalized sphere in the sense of section 2.1).[8]

[6] In February 2015, Agnes Beaudry announced a preprint falsifying the chromatic splitting conjecture at $p=2$ and $n=2$. This is exciting, but also not really a surprise; in the results above, we have had conditions amounting to $p \gg n$, and so perhaps the conjecture will need restatement at small primes.

[7] Many questions about the viability of Picard-graded homotopy groups remain open, all of which seem to be nigh-impossible to answer outside of $n=1$, the most simple case. For instance, we are not even aware of what Pic_2 looks like at the prime 2, nor do we have any idea how to make the assignment $(\lambda \in \operatorname{Pic}(L_{K(2)}\mathsf{Spectra})) \mapsto \pi_\lambda L_{K(2)}\mathbb{S}^0$ "continuous" — this seems to be especially bewildering, actually.

[8] This is the subject of another talk entirely, but a major open question in homotopy theory is Freyd's conjectural "generating hypothesis", which hopes that the functor π_* is faithful when restricted to finite spectra (i.e., a map of finite spectra is nullhomotopic if and only if it induces the zero map on homotopy groups). There is a chromatic approach to this conjecture, envisioned to Devinatz and Hopkins, which begins with analyzing the spectra $\hat{I}\mathbb{S}^0$.

2.3. **Ordinary homology and monochromatic homotopy.** If you know enough algebraic topology to be dangerous, an obvious question unaddressed up to this point is: why not apply homology to this situation? After all, homology is this ultra-computable thing that has been so enormously helpful everywhere else. There is a good reason for avoiding it: a calculation of Ravenel and Wilson says that it is almost devoid of content here.

Theorem (Douglas Ravenel and W. Steve Wilson). *The stable homology groups $K(n)_* H\mathbb{Z}$ vanish. Moreover, this occurs at a finite stage:*

$$K(n)_* K(\mathbb{Z}, q+1) \cong \mathrm{Alt}^q K(n)_* K(\mathbb{Z}, 1+1),$$

where the exterior power is that of Hopf algebras and $K(n)_* K(\mathbb{Z}, 2)$ is generated by n elements as a Hopf algebra. In particular, $K(n)^* K(\mathbb{Z}, q+1)$ is a $\binom{n-1}{q-1}$-dimensional power series ring for $1 \le q \le n$, and it vanishes for $q > n$.[9]

3. Determinantal K-theory

In the remainder of the talk, I want to introduce a new tool in the $K(n)$-local category and to use it to say something about the chromatic splitting conjecture. This comes about by consideration of the two edge cases of the Ravenel–Wilson computation: both of the spaces $K(\mathbb{Z}, 2)$ and $K(\mathbb{Z}, n+1)$ have the property that their $K(n)$-cohomology is a 1-dimensional power series ring. This observation for $K(\mathbb{Z}, 2) = \mathbb{C}\mathrm{P}^\infty$ powers all the computations which are used to get classical complex K-theory off the ground — in fact, though complex K-theory is typically motivated via vector bundles, homotopy theorists have become so adept at studying K-theory that this computation is actually the only observation needed to construct it:

Theorem (Victor Snaith). *There is an equivalence of E_∞-ring spectra*

$$\Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty[\beta^{-1}] \xrightarrow{\simeq} KU,$$

where $\beta : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^\infty$ is the Bott class inducing the isomorphism $H_2 \mathbb{C}\mathrm{P}^1 \to H_2 \mathbb{C}\mathrm{P}^\infty$.

There is an analogous theorem in the $K(n)$-local category for the other edge case:

Theorem (P.). *There is a sequence of spectra XP^n which colimit to $XP^\infty = L_{K(n)} K(\mathbb{Z}, n+1)$ and whose associated-graded consists of generalized sphere spectra. When $p \gg n$, there is an equivalence $XP^1 \simeq \Sigma^{n-n^2} I\mathbb{S}^0$.*[10,11]

The philosophy of this theorem is more interesting than the statement is; the notion that generalized spheres can give more efficient cellular descriptions of chromatically local spectra than just classical cells is fresh and unexplored. In any case, the inclusion $\beta : XP^1 \to XP^\infty$ gives a Picard-graded homotopy class of XP^∞. We define *determinantal K-theory* to be

$$K^{\det} = \left(L_{K(n)} \Sigma^\infty_+ K(\mathbb{Z}, n+1)\right)[\beta^{-1}].$$

Let me now address some applications. Craig Westerland has found an alternative construction of this same ring spectrum, which he uses to prove the following generalization of the calculation of $\pi_* M_1 \mathbb{S}^0$ given earlier:

Theorem (Craig Westerland). *Determinantal K-theory can be constructed in such a way that it acquires a natural map $j^{\det} : K^{\det} \to b\,\mathrm{gl}_1 L_{K(n)} \mathbb{S}^0$, analogous to the classical J-map sending a vector bundle to its associated spherical bundle. Furthermore, when $p \gg n$, one finds a unique factor*

$$\mathbb{Z}_p/(kp) \subseteq [(XP^1)^{\wedge(p-1)k}, L_{K(n)} \mathbb{S}^1]$$

in that Picard–graded homotopy group, along with possibly more p-power torsion of order less than $\mathbb{Z}_p/(kp)$. (In particular, when $n = 1$, XP^1 is simply $\mathbb{S}^2$ and this recovers the classical result of Adams from the start of this talk.)[12]

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[9] There are many strange consequences of this result. One is that the $K(n)$-homology of unstable Postnikov towers must eventually stabilize — and often to something nonzero. Another is that for spectra satisfying chromatic convergence (e.g., all suspension spectra), the chromatic tower of *any* Postnikov section will recover the original spectrum, together with the Postnikov layers you thought you had thrown away.

[10] These $p \gg n$ conditions are quadratic in n.

[11] At $n \ge 2$ and regardless of p, $I\mathbb{S}^0$ can be demonstrated not to be a classical sphere. Instead, it is a choice-free model of the "determinantal sphere", $\mathbb{S}[\det]$.

[12] I've taken care not to mention E-theory or the Morava stabilizer group, but for those in the know: Craig shows that $K^{\det}$ can be expressed as $E_n^{hS\mathbb{S}_n}$, where $S\mathbb{S}_n$ is the kernel of the determinant map $\mathbb{S}_n \to \mathbb{Z}_p^\times$. In particular, this begets a fiber sequence $L_{K(n)}\mathbb{S} \to K^{\det} \xrightarrow{1-\psi^\gamma} K^{\det}$.

Fascinating! Here's another intriguing analogy. First, note that $\mathbb{C}\mathrm{P}^\infty$ has the homotopy type of the classifying space $BU(1)$; the group $U(1)$ has the homotopy type of S^1; and the filtration of $\mathbb{C}\mathrm{P}^\infty$ by finite projective spaces is coincident with the bar filtration on $BU(1)$.

Theorem (P.). *The spectrum $G(1) = \Sigma^{-1} XP^1$ has an A_∞ multiplication which induces equivalences $BG(1)^{(n)} \simeq XP^n$ and $BG(1) \simeq XP^\infty$.*

Noting that $G(1)$ is an invertible spectrum, this means that the $K(n)$-local category has more elements of "Hopf invariant 1" than the classical stable category does.[13]

In addition to β, $K(\mathbb{Z}, n+1)$ comes with another natural class: the fundamental class $\iota_{n+1} : S^{n+1} \to K(\mathbb{Z}, n+1)$. We've expressed $K(\mathbb{Z}, n+1) = XP^\infty$ as a sequential colimit and S^{n+1} is a compact object, so the fundamental class factors through some finite stage XP^m. At the minimal such stage this pushes forward to an interesting class $S^{n+1} \to XP^m \to XP^m / XP^{m-1} = (XP^1)^{\wedge m}$. Craig and I can use the A_∞ result to show that m must equal 1, which begets the following factorizaton:

$$\begin{tikzcd} \mathbb{S}^{n+1} \arrow[r, "\iota_{n+1}"] \arrow[dr, "\alpha"'] & XP^\infty \\ & XP^1. \arrow[u, "\beta"'] \end{tikzcd}$$

In the case $p \gg n$ we have $XP^1 = \Sigma^{n-n^2} I$, and hence α can be interpreted as a class in:

$$\alpha \in [\mathbb{S}^{n+1}, XP^1] = [\mathbb{S}^{n+1}, \Sigma^{n-n^2} I] = [\mathbb{S}^{1+n^2}, I] = \mathrm{Hom}(\pi_{-1-n^2} M_n \mathbb{S}^0, \mathbb{Q}_p/\mathbb{Z}_p).$$

Almost-Theorem (Tyler Lawson, P., and Craig Westerland). *So interpreted, α determines a nonzero and divisible class in $\pi_{-1-n^2} M_n \mathbb{S}^0$.*

This almost-theorem is meant to give meager evidence for the chromatic splitting conjecture: this is precisely where the bottom-most divisible group predicted by the chromatic splitting conjecture *should* live.

[13]Peter May notes that the p-complete category already has more elements of Hopf invariant 1 than the global stable category, so this sort of phenomenon is not new.