

Unoriented Cobordism

Cary Malkiewich

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1 Definition of Cobordism

Let M and N be compact n -dimensional manifolds, with no boundary, not necessarily connected. Let $M \amalg N$ denote their disjoint union. Then we say that M and N are *cobordant* if there exists a compact $(n + 1)$ -dimensional manifold W , whose boundary ∂W is diffeomorphic to $M \amalg N$. Let \mathcal{N}_n denote the set of equivalence classes of compact n -dimensional manifolds (not necessarily connected), where two manifolds are equivalent if they are cobordant in this sense.

\mathcal{N}_n can be made into an abelian group. To define addition, we use either disjoint union or connected sum:

$$[M] + [N] = [M \amalg N] = [M \# N]$$

Next, let \mathcal{N}_* be the graded abelian group whose n th degree component is \mathcal{N}_n :

$$\mathcal{N}_* = \bigoplus_{n=0}^{\infty} \mathcal{N}_n$$

We can think of elements of \mathcal{N}_* as disjoint unions of compact manifolds of varying dimension. Define multiplication on \mathcal{N}_* using the Cartesian product of manifolds:

$$[M][N] = [M \times N]$$

It is easy to check that this is well-defined on equivalence classes and turns \mathcal{N}_* into a commutative graded ring. Next observe that

$$2[M] = [M \amalg M] = [\partial(M \times I)] = 0$$

and so \mathcal{N}_* is a commutative graded algebra over \mathbb{F}_2 . The goal of this note is to outline the proof that \mathcal{N}_* is a polynomial algebra with one generator in each degree not of the form $2^s - 1$:

$$\mathcal{N}_* \cong \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \dots] = \mathbb{F}_2[x_i : i \neq 2^s - 1]$$

2 The Spectrum MO

Let

$$BO(k) = \{k\text{-dimensional subspaces of } \mathbb{R}^\infty\} = \operatorname{colim}_N \left(\frac{O(k+N)}{O(k) \times O(N)} \right)$$

with the usual topology. This space classifies fiber bundles with structure group $O(k)$, including real vector bundles. In other words, it comes with a canonical bundle

$$\gamma^k = \{(x, v) : x \in BO(k), v \in x\}$$

and given any real vector bundle E over a manifold or CW-complex X , there is a bundle map

$$\begin{array}{ccc} E & \longrightarrow & \gamma^k \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BO(k) \end{array}$$

(Recall that a bundle map must be a linear isomorphism on each fiber.) In particular, this gives an isomorphism between E and the pullback $f^*\gamma^k$. The map $f : X \rightarrow BO(k)$ is unique up to homotopy.

Next we recall the construction and properties of Thom spaces. Given a real vector bundle E over X , the Thom space TE is just the one-point compactification of the total space E . It is not hard to check that a map of vector bundles $E \rightarrow F$ yields a based map of Thom spaces $TE \rightarrow TF$, and this construction respects compositions and the identity map, so T is a functor from real vector bundles over topological spaces into based topological spaces.

Define

$$MO(k) = T\gamma^k$$

This yields a sequence of based spaces $\{MO(k)\}_{k=0}^\infty$. To create a spectrum we must give maps

$$S^1 \wedge MO(k) \rightarrow MO(k+1)$$

Here $A \wedge B$ is the smash product of based spaces:

$$(A \times B)/(A \times \{b_0\} \cup \{a_0\} \times B)$$

It is not hard to check that for a vector bundle E and a trivial line bundle ϵ ,

$$T(\epsilon \oplus E) \cong \Sigma TE \cong S^1 \wedge TE$$

To be more precise, these three constructions give naturally isomorphic functors from real vector bundles to based topological spaces.

Next, choose an isomorphism $\mathbb{R} \oplus \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ that inserts the new coordinate in before all the others. Then given a k -dimensional subspace V of \mathbb{R}^∞ , the image of $\mathbb{R} \oplus V \subset \mathbb{R} \oplus \mathbb{R}^\infty$ under our isomorphism gives a $(k+1)$ -dimensional subspace of \mathbb{R}^∞ . This describes a continuous map $BO(k) \rightarrow BO(k+1)$. Next, observe that the pullback of γ^{k+1} under this map is $\epsilon \oplus \gamma^k$:

$$\begin{array}{ccc} \epsilon \oplus \gamma^k & \longrightarrow & \gamma^{k+1} \\ \downarrow & & \downarrow \\ BO(k) & \longrightarrow & BO(k+1) \end{array}$$

Applying the functor T to this map of vector bundles gives

$$T(\epsilon \oplus \gamma^k) \rightarrow T\gamma^{k+1}$$

$$\Sigma MO(k) \rightarrow MO(k+1)$$

This turns our sequence of based spaces $MO(k)$ into a spectrum.

Now we can compose the suspension isomorphisms on homotopy, reduced homology, and reduced cohomology together with these connecting maps:

$$\left\{ \begin{array}{llll} \pi_{n+k}(MO(k)) & \xrightarrow{\sim} & \pi_{n+k+1}(\Sigma MO(k)) & \rightarrow & \pi_{n+k+1}(MO(k+1)) \\ \tilde{H}_{n+k}(MO(k); \mathbb{F}_2) & \xrightarrow{\sim} & \tilde{H}_{n+k+1}(\Sigma MO(k); \mathbb{F}_2) & \rightarrow & \tilde{H}_{n+k+1}(MO(k+1); \mathbb{F}_2) \\ \tilde{H}^{n+k}(MO(k); \mathbb{F}_2) & \xrightarrow{\sim} & \tilde{H}^{n+k+1}(\Sigma MO(k); \mathbb{F}_2) & \leftarrow & \tilde{H}^{n+k+1}(MO(k+1); \mathbb{F}_2) \end{array} \right.$$

and define the homotopy, homology and cohomology groups of MO :

$$\left\{ \begin{array}{ll} \pi_n(MO) & := \operatorname{colim}_k \pi_{n+k}(MO(k)) \\ H_n(MO; \mathbb{F}_2) & := \operatorname{colim}_k \tilde{H}_{n+k}(MO(k); \mathbb{F}_2) \\ H^n(MO; \mathbb{F}_2) & := \lim_k \tilde{H}^{n+k}(MO(k); \mathbb{F}_2) \end{array} \right.$$

(In general this is the correct way to define homotopy and homology, but not cohomology, of a spectrum. In this case we get the same answer, but in general this method has a \lim^1 error term in the cohomology.)

Now we have three graded abelian groups $\pi_*(MO)$, $H_*(MO; \mathbb{F}_2)$, $H^*(MO; \mathbb{F}_2)$. Our next task is to turn the first two into graded rings. To accomplish this we define “multiplication maps” on the spaces $MO(k)$, which turn MO into a “ring spectrum.” Choose an isomorphism $\mathbb{R}^\infty \oplus \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$. Then, as before, this takes k -dimensional subspaces of \mathbb{R}^∞ and ℓ -dimensional subspaces of \mathbb{R}^∞ to a $k + \ell$ -dimensional subspace of \mathbb{R}^∞ , and defines a bundle map

$$\begin{array}{ccc} \gamma^k \times \gamma^\ell & \longrightarrow & \gamma^{k+\ell} \\ \downarrow & & \downarrow \\ BO(k) \times BO(\ell) & \longrightarrow & BO(k+\ell) \end{array}$$

Applying T to the bundles gives

$$\begin{aligned} T(\gamma^k \times \gamma^\ell) &\rightarrow T\gamma^{k+\ell} \\ T\gamma^k \wedge T\gamma^\ell &\rightarrow T\gamma^{k+\ell} \\ MO(k) \wedge MO(\ell) &\rightarrow MO(k+\ell) \end{aligned}$$

These maps give the data of a ring spectrum. They are compatible with the suspension maps in the following sense. If we form a commuting diagram

$$\begin{array}{ccc} \mathbb{R} \oplus \mathbb{R}^\infty \oplus \mathbb{R}^\infty & \longrightarrow & \mathbb{R} \oplus \mathbb{R}^\infty \\ \downarrow & & \downarrow \\ \mathbb{R}^\infty \oplus \mathbb{R}^\infty & \longrightarrow & \mathbb{R}^\infty \end{array}$$

out of our chosen isomorphisms $\mathbb{R} \oplus \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ and $\mathbb{R}^\infty \oplus \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$, then the square does not commute, but the two paths give two linear isomorphisms $\mathbb{R} \oplus \mathbb{R}^\infty \oplus \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ that are homotopic through isomorphisms. These give bundle maps

$$\begin{array}{ccc} \mathbb{R} \oplus \gamma^k \oplus \gamma^\ell & \longrightarrow & \mathbb{R} \oplus \gamma^{k+\ell} \\ \downarrow & & \downarrow \\ \gamma^{k+1} \oplus \gamma^\ell & \longrightarrow & \gamma^{k+\ell+1} \end{array}$$

lying over the homotopy-commuting square

$$\begin{array}{ccc} BO(k) \times BO(\ell) & \longrightarrow & BO(k) \times BO(\ell) \\ \downarrow & & \downarrow \\ BO(k+1) \times BO(\ell) & \longrightarrow & BO(k+\ell+1) \end{array}$$

Applying T yields the homotopy-commuting square

$$\begin{array}{ccc} S^1 \wedge MO(k) \wedge MO(\ell) & \longrightarrow & S^1 \wedge MO(k+\ell) \\ \downarrow & & \downarrow \\ MO(k+1) \wedge MO(\ell) & \longrightarrow & MO(k+\ell+1) \end{array}$$

Now given two maps $S^{n+k} \rightarrow MO(k)$ and $S^{m+\ell} \rightarrow MO(\ell)$, we can use our ring spectrum maps $MO(k) \wedge MO(\ell) \rightarrow MO(k+\ell)$ to get

$$S^{n+k+m+\ell} \cong S^{n+k} \wedge S^{m+\ell} \rightarrow MO(k) \wedge MO(\ell) \rightarrow MO(k+\ell)$$

This construction defines a multiplication

$$\pi_{n+k}(MO(k)) \times \pi_{m+\ell}(MO(\ell)) \rightarrow \pi_{n+k+m+\ell}(MO(k+\ell))$$

which we can check is linear on each factor. The above compatibility condition then implies that

$$\begin{array}{ccc} \pi_{n+k}(MO(k)) \times \pi_{m+\ell}(MO(\ell)) & \longrightarrow & \pi_{n+k+m+\ell}(MO(k+\ell)) \\ \downarrow & & \downarrow \\ \pi_{n+k+1}(MO(k+1)) \times \pi_{m+\ell}(MO(\ell)) & \longrightarrow & \pi_{n+k+m+\ell+1}(MO(k+\ell+1)) \end{array}$$

commutes. Therefore we have a multiplication

$$\pi_n(MO) \times \pi_{m+\ell}(MO(\ell)) \rightarrow \pi_{n+m}(MO)$$

where the first factor and the output are stable, but the second factor is not stable. Finally we check that when $n+k$ is even, the following commutes:

$$\begin{array}{ccc} \pi_{n+k}(MO(k)) \times \pi_{m+\ell}(MO(\ell)) & \longrightarrow & \pi_{n+k+m+\ell}(MO(k+\ell)) \\ \downarrow & & \downarrow \\ \pi_{n+k}(MO(k)) \times \pi_{m+\ell+1}(MO(\ell+1)) & \longrightarrow & \pi_{n+k+m+\ell+1}(MO(k+\ell+1)) \end{array}$$

Therefore this descends to a multiplication

$$\pi_n(MO) \times \pi_m(MO) \rightarrow \pi_{m+n}(MO)$$

This turns $\pi_*(MO)$ into a graded ring. (We have not verified associativity, but this will follow in the next section when we show that it is isomorphic to the commutative graded \mathbb{F}_2 -algebra \mathcal{N}_* .)

Similarly, we use the cross product on homology to define a multiplication

$$\begin{aligned} \tilde{H}_*(MO(k); \mathbb{F}_2) \otimes \tilde{H}_*(MO(\ell); \mathbb{F}_2) &\xrightarrow{\sim} \tilde{H}_*(MO(k) \wedge MO(\ell); \mathbb{F}_2) \\ &\rightarrow \tilde{H}_*(MO(k+\ell); \mathbb{F}_2) \end{aligned}$$

and by carefully choosing representatives with even dimension we get a graded multiplication

$$H_*(MO; \mathbb{F}_2) \otimes H_*(MO; \mathbb{F}_2) \rightarrow H_*(MO; \mathbb{F}_2)$$

turning $H_*(MO; \mathbb{F}_2)$ into a graded ring as well.

3 The Pontryagin-Thom map

The purpose of this section is to describe a graded ring isomorphism

$$\Theta : \mathcal{N}_* \rightarrow \pi_*(MO)$$

Given a compact manifold M , of dimension n , it can be embedded into $(n+k)$ -dimensional Euclidean space, where $k > 0$ depends only on n and not on the particular manifold M of that dimension. (It is not hard to prove that $k = n+1$ works, and with Whitney's doubling trick this can be

refined to $k = n$.) Pick an embedding $M \hookrightarrow \mathbb{R}^{n+k}$ and let ν be the normal bundle of this embedding. That is, ν is a topological space consisting of pairs (m, v) where $m \in M$ is a point on the manifold and $v \in \mathbb{R}^{n+k}$ is a vector perpendicular to M at that point. Since ν is a k -dimensional real vector bundle over M , it is classified by a map $M \rightarrow BO(k)$. By the tubular neighborhood theorem, there is an open neighborhood $N \subset \mathbb{R}^{n+k}$ of M and a diffeomorphism $\nu \cong N$. The closure \bar{N} is an $(n+k)$ -dimensional manifold with boundary ∂N .

With this data, we construct a map of based spaces as follows:

$$S^{n+k} \cong \mathbb{R}^{n+k} \cup \{\infty\} \rightarrow \bar{N}/\partial N \cong T\nu \rightarrow T\gamma^k = MO(k)$$

The homeomorphism $S^{n+k} \cong \mathbb{R}^{n+k} \cup \{\infty\}$ is fixed. The map $\mathbb{R}^{n+k} \cup \{\infty\} \rightarrow \bar{N}/\partial N$ is the identity on $N \subset \mathbb{R}^{n+k}$ and sends everything else to the basepoint of $\bar{N}/\partial N$. The next map $\bar{N}/\partial N \cong T\nu$ is the one-point compactification of the chosen diffeomorphism $N \cong \nu$. The final map $T\nu \rightarrow T\gamma^k$ is the one-point compactification of the map $\nu \rightarrow \gamma^k$ that classifies the normal bundle ν . The composite of all these maps is a map $S^{n+k} \rightarrow MO(k)$.

This construction uses arbitrary choices, and different choices lead to different maps $S^{n+k} \rightarrow MO(k)$. If k is sufficiently large, however, then all such maps are homotopic. Therefore we get a well-defined element of $\pi_{n+k}(MO(k))$ without having to choose anything other than the integer k . Ridding ourselves of even this choice, we pass to the limit and get a well-defined element $\Theta(M) \in \pi_n(MO)$.

Before showing that Θ is well-defined on cobordism classes, we will show that it is a homomorphism:

$$\Theta(M \amalg M') = \Theta(M) + \Theta(M')$$

This is relatively easy. Choose an embedding of $M \amalg M'$ into \mathbb{R}^{n+k} so that M and its tubular neighborhood are confined to the top half $\mathbb{R}^{n+k-1} \times \mathbb{R}_+$ and M' and its tubular neighborhood are confined to the bottom half $\mathbb{R}^{n+k-1} \times \mathbb{R}_-$. Then the map $S^{n+k} \rightarrow MO(k)$ that we construct using this embedding factors as

$$S^{n+k} \rightarrow S^{n+k} \vee S^{n+k} \rightarrow \bar{N}/\partial N \vee \bar{N}'/\partial N' \cong T\nu \vee T\nu' \rightarrow MO(k)$$

This describes a sum in the group $\pi_{n+k}(MO(k))$ of two separate maps, and if we trace through their geometric descriptions we see that the two maps

are in fact the Pontryagin-Thom maps for M and M' . This establishes

$$\Theta(M \amalg M') = \Theta(M) + \Theta(M')$$

Next, given a cobordism W between M and M' , embed W into a Euclidean half-space $\mathbb{R}^{n+k} \times \mathbb{R}_+$ such that $\partial W \cong M \amalg M'$ is embedded into $\mathbb{R}^{n+k} \times \{0\}$. Then walk through the Pontryagin-Thom construction again, this time taking a tubular neighborhood of W whose restriction to $\mathbb{R}^{n+k} \times \{0\}$ is a tubular neighborhood of M and M' :

$$D^{n+k+1} \cong \mathbb{R}^{n+k} \times \mathbb{R}_+ \cup \{\infty\} \rightarrow \bar{N}/\partial N \rightarrow MO(k)$$

This gives a map $D^{n+k+1} \rightarrow MO(k)$ whose restriction to S^{n+k} is the map constructed from $M \amalg M'$. Therefore $\Theta(M \amalg M') = 0$, so $\Theta(M) = \Theta(M')$. Therefore Θ gives a graded group homomorphism

$$\Theta : \mathcal{N}_* \rightarrow \pi_*(MO)$$

injectivity

surjectivity

ring map

4 Cohomology and Homology of BO and MO

For the rest of the paper, all homology and cohomology will be taken with \mathbb{F}_2 coefficients.

Our goal in this section is to compute the cohomology and homology of the space BO , and the cohomology and homology of the spectrum MO :

$$H^*(BO) \cong \mathbb{F}_2[w_1, w_2, w_3, \dots] \text{ as rings,} \quad |w_i| = i$$

$$H^*(MO) \cong \mathbb{F}_2[w_1, w_2, w_3, \dots] \text{ as groups,} \quad |w_i| = i$$

$$H_*(BO) \cong \mathbb{F}_2[x_1, x_2, x_3, \dots] \text{ as rings,} \quad |x_i| = i$$

$$H_*(MO) \cong \mathbb{F}_2[x_1, x_2, x_3, \dots] \text{ as rings,} \quad |x_i| = i$$

The Thom isomorphism is responsible for the top two being isomorphic as groups, and for the bottom two being isomorphic as rings. We begin with a review of the Thom isomorphism and Stiefel-Whitney classes.

Given an n -dimensional real vector bundle $E \rightarrow B$, we can take the fiberwise one-point compactification to get a sphere bundle \bar{E} with a section. Applying the Serre spectral sequence, we compute the cohomology of this sphere bundle in terms of the cohomology of B . Collapsing the chosen section down to a point gives the Thom space TE of the original vector bundle. This calculation yields the Thom isomorphism theorem:

$$H^k(B) \cong \tilde{H}^{n+k}(TE)$$

(Keeping in mind that coefficients are in \mathbb{F}_2 , we do not need to assume that the bundle is oriented.)

To describe the isomorphism explicitly, we give the bundle E a (fiberwise) metric and notice that $TE \cong DE/SE$, where DE is the unit disc bundle and SE is the unit sphere bundle. Then compose

$$H^k(B) \rightarrow H^k(DE) \rightarrow H^{n+k}(DE, SE) \xrightarrow{\sim} \tilde{H}^{n+k}(TE)$$

where the first map comes from $DE \rightarrow B$, the second map is the cup product with the Thom class U , and the last isomorphism comes from excision. The Thom class $U \in H^n(DE, SE)$ is characterized by the fact that its restriction to the fiber (D^n, S^{n-1}) is the generator of $H^n(D^n, S^{n-1})$.

Since $MO(k) = T\gamma^k$ is a Thom space over $BO(k)$, we get an isomorphism

$$\varphi_k : H^i(BO(k)) \rightarrow \tilde{H}^{i+n}(MO(k))$$

in which the main step is cup product with the Thom class $U_k \in \tilde{H}^k(D\gamma^k, S\gamma^k)$. Dualizing the above construction, there is an isomorphism on homology

$$\psi_k : H_i(BO(k)) \leftarrow \tilde{H}_{i+n}(MO(k))$$

in which the main step is cap product with the Thom class.

The next ingredient we need are Stiefel-Whitney classes. Using the Thom isomorphism together with the Steenrod squares $\text{Sq}^i : H^n(X, A; \mathbb{F}_2) \rightarrow H^{n+i}(X, A; \mathbb{F}_2)$, we can define

$$w_i = \varphi_k^{-1} \text{Sq}^i U = \varphi_k^{-1} \text{Sq}^i \varphi_k(1) \in H^i(BO(k))$$

Since $BO(k)$ classifies k -dimensional real vector bundles, every real vector bundle $E \rightarrow X$ gives a unique homotopy class of maps in $[X, BO(k)]$. Any map in this homotopy class induces the same map $H^i(BO(k)) \rightarrow H^i(X)$. The image of w_i under this map is $w_i(E)$, the i th Stiefel-Whitney class of the vector bundle E . The axioms for the Steenrod squares (given in section 5) translate easily into the axioms for the characteristic classes w_i :

- $w_0 = 1$
- $w_1(\gamma^1) \neq 0$
- $w_i = 0$ when $i > \dim E$
- If $w = \sum_i w_i \in H^*(B)$, then for two vector bundles $E_1 \rightarrow B_1$ and $E_2 \rightarrow B_2$, we have $w(E_1 \times E_2) = w(E_1)w(E_2)$ in $H^*(B_1 \times B_2)$

Next, we calculate

$$H^*(BO(n)) = \mathbb{F}_2[w_1, w_2, \dots, w_n]$$

using induction on n . When $n = 0$ the statement is trivial. Inductively, if $H^*(BO(n-1))$ is given by the above formula, then using the Gysin sequence for $BO(n)$ we can deduce that $H^*(BO(n))$ is generated by w_1 through w_n . Therefore

$$H^*(BO(n)) = \mathbb{F}_2[w_1, w_2, \dots, w_n]/I$$

To show that $I = 0$, we consider the map classifying the product of n copies of γ^1 :

$$\begin{aligned} \gamma^1 \times \gamma^1 \times \dots \times \gamma^1 &\rightarrow \gamma^n \\ BO(1)^n &\rightarrow BO(n) \end{aligned}$$

The induced map on cohomology takes $w_i \in H^i(BO(n))$ to $w_i((\gamma^1)^{\times n})$. To calculate this we start with

$$w(\gamma^1) = 1 + w_1$$

and then describe

$$H^*(BO(1)^n) \cong H^*(BO(1))^{\otimes n} \cong (\mathbb{F}_2[\alpha])^{\otimes n} \cong \mathbb{F}_2[\alpha_1, \alpha_2, \dots, \alpha_n]$$

Here α_k is a degree one cohomology class generating the cohomology of the k th copy of $BO(1)$. Therefore the k th copy of γ^1 in the product has $w(\gamma_k^1) = 1 + \alpha_k$. The product rule for Stiefel-Whitney classes gives

$$w\left(\prod_k \gamma_k^1\right) = \prod_k w(\gamma_k^1) = \prod_k (1 + \alpha_k)$$

Therefore $w_i(\prod_k \gamma_k^1)$ is the i th symmetric polynomial σ_i in the α_k . By naturality of w_i we get

$$\begin{aligned} H^*(BO(n)) &\rightarrow H^*(BO(1))^{\otimes n} \\ \mathbb{F}_2[w_1, w_2, \dots, w_n] &\rightarrow \mathbb{F}_2[w_1, w_2, \dots, w_n]/I \rightarrow \mathbb{F}_2[\alpha_1, \dots, \alpha_n] \\ w_i &\mapsto \sigma_i(\alpha_1, \dots, \alpha_n) \end{aligned}$$

It can be shown using basic algebra that the polynomials σ_i are algebraically independent. Therefore the above map is injective. Therefore $I = 0$ and

$$H^*(BO(n)) = \mathbb{F}_2[w_1, w_2, \dots, w_n]$$

Recall that $BO = \operatorname{colim}_n BO(n)$ under the maps $BO(n) \rightarrow BO(n+1)$. It can be given the structure of a CW complex in which each $BO(n)$ is a subcomplex. Therefore every chain in BO comes from some $BO(n)$, so

$$H_*(BO) \cong \lim_{\rightarrow} H_*(BO(n))$$

Using the natural pairing between homology and cohomology one can show that in this case

$$H^*(BO) \cong \lim_{\leftarrow} H^*(BO(n)) \cong \mathbb{F}_2[w_1, w_2, w_3, \dots]$$

This finishes the first of our four calculations. To get the next one, $H^*(MO)$, we construct the “stable Thom isomorphism”

$$\varphi : H^*(BO) \rightarrow H^*(MO)$$

This exists simply because the Thom isomorphism makes the following square commute:

$$\begin{array}{ccc} H^*(MO(k)) & \longleftarrow & H^*(MO(k+1)) \\ \uparrow & & \uparrow \\ H^*(BO(k)) & \longleftarrow & H^*(BO(k+1)) \end{array}$$

However, we have said nothing about the cup product, so we only get

$$H^*(MO) = \mathbb{F}_2[w_1, w_2, w_3, \dots], \quad |w_i| = i$$

as groups.

Since \mathbb{F}_2 is a field, the universal coefficient theorem tells us that $H_n(BO(k))$ is naturally isomorphic to the dual of $H^n(BO(k))$ and vice-versa. In the limit, this gives a natural isomorphism between $H_n(BO)$ and the dual of $H^n(BO)$. This isomorphism is given by the natural pairing between homology and cohomology, which we will denote by brackets $\langle \cdot, \cdot \rangle$. Similarly, we get a natural perfect pairing between $\tilde{H}_{n+k}(MO(k))$ and $\tilde{H}^{n+k}(MO(k))$, which passes to a natural perfect pairing between $H_n(MO)$ and $H^n(MO)$.

We have determined the additive structure of $H^*(BO) \cong H^*(MO)$; dualizing gives us the additive structure of $H_*(BO) \cong H_*(MO)$. We finish this section by calculating the multiplicative structure of these homology groups. First observe that, under $MO(k) \wedge MO(\ell) \rightarrow MO(k + \ell)$, the Thom class of $MO(k + \ell)$ pulls back to the Thom class on $MO(k) \wedge MO(\ell)$. (This latter space is the Thom space of $\gamma^k \times \gamma^\ell$ over $BO(k) \times BO(\ell)$.)

$$\begin{array}{ccccc} \tilde{H}_*(MO(k)) \otimes \tilde{H}_*(MO(\ell)) & \longrightarrow & \tilde{H}_*(MO(k) \wedge MO(\ell)) & \longrightarrow & \tilde{H}_*(MO(k + \ell)) \\ \downarrow & & \downarrow & & \downarrow \\ H_*(BO(k)) \otimes H_*(BO(\ell)) & \longrightarrow & H_*(BO(k) \times BO(\ell)) & \longrightarrow & H_*(BO(k + \ell)) \end{array}$$

The downward maps are capping with the relevant Thom class. The first square commutes using a standard identity relating cup and cap products. The second square commutes by the naturality of the Thom class and the cap product. Therefore the whole square commutes. Taking the limit of this square gives

$$\begin{array}{ccc} H_*(MO) \otimes H_*(MO) & \longrightarrow & H_*(MO) \\ \downarrow & & \downarrow \\ H_*(BO) \otimes H_*(BO) & \longrightarrow & H_*(BO) \end{array}$$

which implies that the Thom isomorphism $H_*(MO) \rightarrow H_*(BO)$ is a ring isomorphism. Now we just need the ring $H_*(BO)$.

We have seen in this section that the map $BO(1)^n \rightarrow BO(n)$ gives an injective map on cohomology in every degree. We have also seen that $BO(n) \rightarrow BO$ gives an injective map on cohomology in degrees 0 through n . Therefore $BO(1)^n \rightarrow BO(n) \rightarrow BO$ is injective on cohomology through

degree n . Dualizing, the map on homology is surjective in degree 0 through n . To describe its image, we first calculate

$$H_*(BO(1)) = H_*(\mathbb{RP}^\infty) \cong \mathbb{F}_2\langle x_0, x_1, x_2, \dots \rangle$$

By this we mean that the x_i form an additive basis. The degree of x_i is i . By abuse of notation, let $x_i \in H_i(BO)$ be the image of $x_i \in H_i(BO(1))$. Then $x_0 \in H_0(BO)$ is simply 1.

Recall that the multiplication on $H_*(BO)$ was defined as follows: to multiply $\bar{\alpha} \in H_*(BO)$ coming from $\alpha \in H_*(BO(k))$ with $\bar{\beta} \in H_*(BO)$ coming from $H_*(BO(\ell))$, we take the image of $\alpha \otimes \beta \in H_*(BO(k) \times BO(\ell))$ under the maps

$$BO(k) \times BO(\ell) \rightarrow BO(k + \ell) \rightarrow BO$$

Iterating this process n times, this tells us that the map

$$H_*(BO(1))^{\otimes n} \cong \mathbb{F}_2\langle x_0, x_1, x_2, \dots \rangle^{\otimes n} \rightarrow H_*(BO(n)) \rightarrow H_*(BO)$$

gives all n -fold multiplications in $H_*(BO)$ of elements coming from $H_*(BO(1))$. More precisely, it takes $x_{i_1} \otimes x_{i_2} \otimes \dots \otimes x_{i_n}$ to $x_{i_1}x_{i_2}\dots x_{i_n}$.

The map $BO(1)^n \rightarrow BO(n)$ induces a map on cohomology whose image is symmetric polynomials. Therefore the map on cohomology commutes with the action of the symmetric group Σ_n on $BO(1)^n$. Dualizing, this implies that the map on homology also commutes with the action of Σ_n . Therefore $x_{i_1} \otimes \dots \otimes x_{i_n}$ has the same image as $x_{i_{\sigma(1)}} \otimes \dots \otimes x_{i_{\sigma(n)}}$ for every permutation $\sigma \in \Sigma_n$. Therefore $x_{i_1} \dots x_{i_n} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(n)}}$ in the ring $H_*(BO)$. So the generators x_i coming from $H_*(BO(1))$ all commute with each other.

Let $M \subset \mathbb{F}_2\langle x_0, x_1, x_2, \dots \rangle^{\otimes n}$ be generated by all the elementary tensors $x_{i_1} \otimes \dots \otimes x_{i_n}$ such that the i_n are non-decreasing. Since the x_i commute in $H_*(BO)$, the composite $M \rightarrow H_*(BO(1))^n \rightarrow H_*(BO)$ is still surjective. In degree k , the dimension of M is the number of unordered partitions of k into positive integers. Call this number $f(k)$. Then $f(k)$ is also the dimension of

$$H^*(BO) \cong \mathbb{F}_2[w_1, w_2, w_3, \dots]$$

in degree k . By duality, $H_k(BO)$ has dimension $f(k)$ as well. In degrees $0 \leq k \leq n$, the composite $M \rightarrow H_*(BO(1))^n \rightarrow H_*(BO)$ is a surjective map between spaces of dimension $f(k)$. Therefore it is an isomorphism in these degrees.

The effect of all this is that we have defined a map of rings

$$\mathbb{F}_2[x_1, x_2, x_3, \dots] \rightarrow M \subset \mathbb{F}_2\langle x_0, x_1, x_2, \dots \rangle^{\otimes n} \rightarrow H_*(BO)$$

only in degrees 0 through n , but in these degrees it is an isomorphism. If we increment n , then we get the new composite map

$$\mathbb{F}_2[x_1, x_2, x_3, \dots] \rightarrow M_{n+1} \subset \mathbb{F}_2\langle x_0, x_1, x_2, \dots \rangle^{\otimes(n+1)} \rightarrow H_*(BO)$$

which extends the previous map to an isomorphism through degree $n + 1$. Therefore these maps patch together to give an isomorphism

$$\mathbb{F}_2[x_1, x_2, x_3, \dots] \rightarrow H_*(BO)$$

in all degrees. Using Thom isomorphism this gives

$$H_*(MO) \cong \mathbb{F}_2[x_1, x_2, x_3, \dots]$$

as rings, with the multiplication coming from

$$MO(1)^{\wedge n} \rightarrow MO(n)$$

This finishes our four calculations.

5 The Steenrod Squares on $\mathbb{R}\mathbb{P}^\infty$

We continue to take all homology and cohomology with \mathbb{F}_2 coefficients.

Recall that the Steenrod squares $\text{Sq}^i : H^n(X, A; \mathbb{F}_2) \rightarrow H^{n+i}(X, A; \mathbb{F}_2)$ for $i \geq 0$ are natural homomorphisms satisfying the following properties:

- Sq^0 is the identity map
- Sq^1 is the Bockstein homomorphism arising from $0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0$
- $\text{Sq}^i(x) = x^2$ when $i = |x|$
- $\text{Sq}^i(x) = 0$ when $i > |x|$
- Sq^i commutes with the connecting homomorphism in the long exact sequence on cohomology (in particular, it commutes with the suspension isomorphism on reduced cohomology)

- $\text{Sq}^n(x \cup y) = \sum_{i+j=n} \text{Sq}^i(x) \cup \text{Sq}^j(y)$ (the Cartan formula)
- $\text{Sq}^a \text{Sq}^b = \sum_c \binom{b-c-1}{a-2c} \text{Sq}^{a+b-c} \text{Sq}^c$ when $a < 2b$ (the Adem relations)

The Steenrod algebra \mathcal{A} is the free \mathbb{F}_2 -algebra on $1 = \text{Sq}^0, \text{Sq}^1, \text{Sq}^2, \dots$ modulo the Adem relations. Then every element of \mathcal{A} gives a well-defined natural operation $H^*(X; \mathbb{F}_2) \rightarrow H^*(X; \mathbb{F}_2)$, using addition in \mathcal{A} to denote addition of homomorphisms and multiplication in \mathcal{A} to denote composition of homomorphisms. \mathcal{A} is not a commutative algebra, and we will write composition from right to left.

If X is any space then the action of the Steenrod squares yields a bilinear map $\mathcal{A} \times H^*(X) \rightarrow H^*(X)$, which corresponds to a linear map $\mathcal{A} \otimes H^*(X) \rightarrow H^*(X)$. We will denote this map simply by \cdot .

If $I = (i_0, i_1, \dots, i_r)$ is a sequence of positive integers, then we write $\text{Sq}^I = \text{Sq}^{i_0} \text{Sq}^{i_1} \dots \text{Sq}^{i_r}$. Then \mathcal{A} is additively generated by Sq^I as I ranges over all finite sequences of positive integers and the sequence (0) . The Adem relations imply that we can restrict our attention to sequences I such that $i_j \geq 2i_{j+1}$ for all j . These sequences are called *admissible*. So \mathcal{A} is generated additively by Sq^I as I ranges over all admissible sequences. In fact, these monomials form an additive basis for \mathcal{A} , which we refer to as the *Serre-Cartan basis*.

The Steenrod squares have a comultiplication $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ given by

$$\Delta(\text{Sq}^n) = \sum_{i+j=n} \text{Sq}^i \otimes \text{Sq}^j$$

and extended to the rest of \mathcal{A} as a ring homomorphism. The Cartan formula can be rewritten as

$$\text{Sq}^n(x \cup y) = (\Delta \text{Sq}^n)(x \otimes y)$$

which shows that the following square commutes:

$$\begin{array}{ccc}
\mathcal{A} \otimes H^*(X) \otimes H^*(X) & \xrightarrow{1 \otimes \cup} & \mathcal{A} \otimes H^*(X) \\
\downarrow \Delta \otimes 1 \otimes 1 & & \downarrow \cdot \\
\mathcal{A} \otimes \mathcal{A} \otimes H^*(X) \otimes H^*(X) & & \\
\downarrow 1 \otimes \tau \otimes 1 & & \\
\mathcal{A} \otimes H^*(X) \otimes \mathcal{A} \otimes H^*(X) & & \\
\downarrow \cdot \otimes \cdot & & \\
H^*(X) \otimes H^*(X) & \xrightarrow{\cup} & H^*(X)
\end{array}$$

Let \mathcal{A}^* be the graded dual of \mathcal{A} as a vector space over \mathbb{F}_2 . (In other words, \mathcal{A}^* is defined to be a graded vector space whose n th graded component is the dual of the n th graded component of \mathcal{A} .) Then the dual of the comultiplication map

$$\mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$$

is a map of graded vector spaces the form

$$\mathcal{A}^* \otimes \mathcal{A}^* \rightarrow \mathcal{A}^*$$

and therefore it defines a graded multiplication on \mathcal{A}^* , turning it into a graded algebra over \mathbb{F}_2 . We will assume the following result: if we take the basis on \mathcal{A}^* that is dual to the Serre-Cartan basis on \mathcal{A} , and we let $\xi_i \in \mathcal{A}^*$ be the basis element dual to $\text{Sq}^{(2^k, 2^{k-1}, \dots, 4, 2, 1)}$, then \mathcal{A}^* is isomorphic to the free \mathbb{F}_2 -algebra on the ξ_i :

$$\mathcal{A}^* = \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \dots], \quad |\xi_i| = 2^i - 1$$

In particular, \mathcal{A}^* is a commutative algebra.

Recall next that $K(\mathbb{F}_2, n)$ is a CW-complex such that

$$\pi_i(K(\mathbb{F}_2, n)) = \begin{cases} \mathbb{F}_2 & i = n \\ 0 & i \neq n \end{cases}$$

It follows that $H^n(K(\mathbb{F}_2, n); \mathbb{F}_2) = \mathbb{F}_2$.

Let $\iota_n \in H^n(K(\mathbb{F}_2, n); \mathbb{F}_2)$ be the nontrivial element. Then there is a natural map of sets $[X, K(\mathbb{F}_2, n)] \rightarrow H^n(X; \mathbb{F}_2)$ that takes the image of ι_n in $H^n(X; \mathbb{F}_2)$ under a given map $X \rightarrow K(\mathbb{F}_2, n)$. By the Brown Representability Theorem, this map is a bijection.

Using the Serre Spectral Sequence one can compute the cohomology of the Eilenberg-MacLane spaces $K(\mathbb{F}_2, n)$ and obtain the following:

$$H^*(K(\mathbb{F}_2, n)) = \mathbb{F}_2[\text{Sq}^I \iota_n : I \text{ admissible, } e(I) < n]$$

Here $e(I) = \sum_j i_j - 2i_{j+1}$ is the *excess* of I . When I is admissible and nonzero, the terms of the excess are all nonnegative, and the last term of the excess is positive.

In particular, for $\mathbb{R}\mathbb{P}^\infty = K(\mathbb{F}_2, 1)$ we get the familiar result

$$H^*(\mathbb{R}\mathbb{P}^\infty; \mathbb{F}_2) = \mathbb{F}_2[\iota_1] = \mathbb{F}_2[\alpha]$$

We can easily compute the action of \mathcal{A} on this group using the axioms for the Steenrod squares. If we let $\text{Sq} = \text{Sq}^0 + \text{Sq}^1 + \dots$, then the Cartan formula implies that $\text{Sq} : H^*(X; \mathbb{F}_2) \rightarrow H^*(X; \mathbb{F}_2)$ is a ring homomorphism. The squaring axiom gives

$$\text{Sq}^1 \alpha = \alpha^2$$

Therefore

$$\text{Sq} \alpha = \alpha + \alpha^2$$

Since Sq is a ring homomorphism,

$$\text{Sq}(\alpha^n) = (\alpha + \alpha^2)^n = \sum_i \binom{n}{i} \alpha^{n+i}$$

Picking out the terms of each degree gives

$$\text{Sq}^i \alpha^n = \binom{n}{i} \alpha^{n+i}$$

Setting $n = 2^k$ gives

$$\text{Sq}(\alpha^{2^k}) = \sum_i \binom{2^k}{i} \alpha^{2^k+i} = \alpha^{2^k} + \alpha^{2^k+1}$$

Therefore

$$\text{Sq}^I \alpha = \begin{cases} \alpha^{2^k+1} & I = (2^k, 2^{k-1}, \dots, 4, 2, 1) \\ 0 & \text{otherwise} \end{cases}$$

6 The Steenrod Squares on MO

Recall that all homology and cohomology is taken with \mathbb{F}_2 coefficients.

Define $Sq^i : H^n(MO) \rightarrow H^{n+i}(MO)$ by taking a representative inside $\widetilde{H}^{n+k}(MO(k))$, applying Sq^i to land in $\widetilde{H}^{n+k+i}(MO(k))$, and passing to the limit in $H^{n+i}(MO)$. This is well-defined because the Steenrod squares commute with suspension isomorphisms and maps of spaces, and these maps connect the different representatives together. This turns $H^n(MO)$ into a module over \mathcal{A} .

The \mathcal{A} -module structure on $H^n(MO)$ yields a \mathbb{F}_2 -linear map

$$\mathcal{A} \otimes_{\mathbb{F}_2} H^*(MO) \rightarrow H^*(MO)$$

Applying the graded dual to this map yields

$$\psi : H_*(MO) \rightarrow \mathcal{A}^* \otimes_{\mathbb{F}_2} H_*(MO)$$

which we call a “coaction” of \mathcal{A}^* on $H_*(MO)$. From here out we drop the \mathbb{F}_2 from the tensor products as well.

Our next goal is to show that ψ is a homomorphism of graded rings. (Multiplication in $\mathcal{A}^* \otimes H_*(MO)$ is defined in the usual way. There is no ambiguity as to what the twist map $A \otimes B \rightarrow B \otimes A$ is, since we are over \mathbb{F}_2 and so there are no signs.) To establish this we check that this diagram commutes:

$$\begin{array}{ccc}
 H_*(MO) \otimes H_*(MO) & \xrightarrow{\mu_*} & H_*(MO) \\
 \downarrow \psi \otimes \psi & & \downarrow \psi \\
 \mathcal{A}^* \otimes H_*(MO) \otimes \mathcal{A}^* \otimes H_*(MO) & & \\
 \downarrow 1 \otimes \tau \otimes 1 & & \\
 \mathcal{A}^* \otimes \mathcal{A}^* \otimes H_*(MO) \otimes H_*(MO) & \xrightarrow{\Delta^* \otimes \mu_*} & \mathcal{A}^* \otimes H_*(MO)
 \end{array}$$

It suffices to check this diagram, then take the limit and dualize:

$$\begin{array}{ccc}
\tilde{H}^*(MO(k)) \otimes \tilde{H}^*(MO(\ell)) & \xleftarrow{\mu_{k,\ell}^*} & H^*(MO(k+\ell)) \\
\uparrow \cdot \otimes \cdot & & \uparrow \cdot \\
\mathcal{A} \otimes \tilde{H}^*(MO(k)) \otimes \mathcal{A} \otimes \tilde{H}^*(MO(\ell)) & & \\
\uparrow 1 \otimes \tau \otimes 1 & & \\
\mathcal{A} \otimes \mathcal{A} \otimes \tilde{H}^*(MO(k)) \otimes \tilde{H}^*(MO(\ell)) & \xleftarrow{\Delta \otimes \mu_{k,\ell}^*} & \mathcal{A} \otimes H^*(MO(k+\ell))
\end{array}$$

This diagram splits into two squares:

$$\begin{array}{ccc}
\tilde{H}^*(MO(k)) \otimes \tilde{H}^*(MO(\ell)) & \xrightarrow{\cup} & \tilde{H}^*(MO(k) \wedge MO(\ell)) \\
\uparrow & & \uparrow \\
\mathcal{A} \otimes \tilde{H}^*(MO(k)) \otimes \mathcal{A} \otimes \tilde{H}^*(MO(\ell)) & & \\
\uparrow 1 \otimes \tau \otimes 1 & & \\
\mathcal{A} \otimes \mathcal{A} \otimes \tilde{H}^*(MO(k)) \otimes \tilde{H}^*(MO(\ell)) & & \\
\uparrow \Delta \otimes 1 \otimes 1 & & \\
\mathcal{A} \otimes \tilde{H}^*(MO(k)) \otimes \tilde{H}^*(MO(\ell)) & \xrightarrow{1 \otimes \cup} & \mathcal{A} \otimes H^*(MO(k) \wedge MO(\ell))
\end{array}$$

on the left, which commutes by the Cartan formula, followed by

$$\begin{array}{ccc}
\tilde{H}^*(MO(k) \wedge MO(\ell)) & \xleftarrow{\mu_{k,\ell}^*} & H^*(MO(k+\ell)) \\
\uparrow & & \uparrow \\
\mathcal{A} \otimes \tilde{H}^*(MO(k) \wedge MO(\ell)) & \xleftarrow{1 \otimes \mu_{k,\ell}^*} & \mathcal{A} \otimes H^*(MO(k+\ell))
\end{array}$$

on the right, which commutes by naturality of the Steenrod squares. This finishes the proof that $\psi : H_*(MO) \rightarrow \mathcal{A}^* \otimes H_*(MO)$ is a map of rings.

Recall that $H_*(MO) \cong \mathbb{F}_2[x_1, x_2, \dots]$ where $|x_i| = i$. Define

$$N_* = H_*(MO) / \{x_{2^s-1}\}_{s=1}^\infty \cong \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \dots]$$

Dualizing the quotient map $p : H_*(MO) \rightarrow N_*$ gives an injective map $q : N^* \rightarrow H^*(MO)$. This realizes N^* as an \mathbb{F}_2 -subspace of $H^*(MO)$. I

claim that $H^*(MO)$ is a free \mathcal{A} -module generated by this subspace, i.e. the composite

$$\mathcal{A} \otimes N^* \xrightarrow{1 \otimes q} \mathcal{A} \otimes H^*(MO) \xrightarrow{\cdot} H^*(MO)$$

is an isomorphism of graded vector spaces. To prove this it suffices to prove that the graded dual map

$$H_*(MO) \xrightarrow{\psi} \mathcal{A}^* \otimes H_*(MO) \xrightarrow{1 \otimes p} \mathcal{A}^* \otimes N_*$$

is an isomorphism of graded vector spaces. This is made easier by the fact that it is a map of rings: we only need to examine what happens to the generators.

Recall from an earlier section that

$$H_*(MO) \cong H_*(BO) \cong \mathbb{F}_2[x_1, x_2, x_3, \dots]$$

where x_j is the image of the nonzero element of $H_j(BO(1))$ in $H_j(BO)$. Through Thom isomorphism, it is the image of the nonzero element of $\tilde{H}_{j+1}(MO(1))$ in $H_j(MO)$. Therefore we examine the map

$$\tilde{H}_{*+1}(MO(1)) \xrightarrow{\psi} \mathcal{A}^* \otimes H_{*+1}(MO(1))$$

which by definition passes to ψ on $H_*(MO)$. Now

$$MO(1) \cong D\gamma^1/S\gamma^1 = D\gamma^1/S^\infty \simeq D\gamma^1 \simeq \mathbb{R}\mathbb{P}^\infty$$

Setting

$$\tilde{H}^*(MO(1)) \cong \tilde{H}^*(\mathbb{R}\mathbb{P}^\infty) \cong \mathbb{F}_2\langle \alpha, \alpha^2, \dots \rangle$$

we see that α^{j+1} is the only nonzero element of $\tilde{H}^{j+1}(MO(1))$, so it must be the only thing that pairs with $x_j \in \tilde{H}_{j+1}(MO(1))$ to give 1. Therefore $\langle x_j, \alpha^{j+1} \rangle = 1$ and all other pairings are zero for degree reasons.

We can now calculate one term of ψ on $H_*(MO(1))$:

$$\begin{aligned} \langle \text{Sq}^J \otimes \alpha, \psi(x_i) \rangle &= \langle \text{Sq}^J \alpha, x_i \rangle \\ &= \begin{cases} \langle \alpha^{2^\ell}, x_i \rangle & J = (2^{\ell-1}, \dots, 4, 2, 1) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & i = 2^\ell - 1, J = (2^{\ell-1}, \dots, 4, 2, 1) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Therefore in $\tilde{H}_{*+1}(MO(1))$, $\psi(x_i)$ is an element of $\mathcal{A}^* \otimes \tilde{H}_{*+1}(MO(1))$ which pairs with $Sq^J \otimes \alpha$ to give 1 only when $i = 2^\ell - 1$ and $J = (2^{\ell-1}, \dots, 4, 2, 1)$. Pushing this forward to $H_*(MO)$, $\psi(x_i)$ is an element of $\mathcal{A}^* \otimes H_*(MO)$ which pairs with $Sq^J \otimes 1$ to give 1 only when $i = 2^\ell - 1$ and $J = (2^{\ell-1}, \dots, 4, 2, 1)$. Therefore

$$\psi(x_i) = \begin{cases} \xi_\ell \otimes 1 + \dots & i = 2^\ell - 1 \\ 0 + \dots & \text{otherwise} \end{cases}$$

plus terms whose second factor has degree > 0 .

The unit of \mathcal{A} acts as the identity on cohomology. In diagrams, this means that the composition

$$H^*(MO) \cong \mathbb{F}_2 \otimes H^*(MO) \rightarrow \mathcal{A} \otimes H^*(MO) \rightarrow H^*(MO)$$

is the identity map. Dualizing gives

$$H_*(MO) \xrightarrow{\psi} \mathcal{A}^* \otimes H_*(MO) \rightarrow \mathbb{F}_2 \otimes H_*(MO) \cong H_*(MO)$$

where the map $\mathcal{A}^* \rightarrow \mathbb{F}_2$ is an isomorphism on degree 0 and the zero map on all higher degrees. Therefore

$$\psi(x_i) = 1 \otimes x_i + \dots$$

plus terms whose first factor has degree > 0 .

Combining these two results gives

$$\psi(x_i) = \begin{cases} \xi_\ell \otimes 1 + \dots + 1 \otimes x_i & J = (2^{\ell-1}, \dots, 4, 2, 1) \\ 0 + \dots + 1 \otimes x_i & \text{otherwise} \end{cases}$$

plus terms whose two factors both have positive degree. Using the simple description of the quotient map

$$p : \mathbb{F}_2[x_1, x_2, x_3, x_4, x_5, \dots] \rightarrow \mathbb{F}_2[x_2, x_4, x_5, \dots]$$

we get

$$((1 \otimes p) \circ \psi)(x_i) = \begin{cases} \xi_\ell \otimes 1 + \dots + 0 & i = 2^\ell - 1 \\ 0 + \dots + 1 \otimes x_i & \text{otherwise} \end{cases}$$

Therefore the map

$$\mathbb{F}_2[x_1, x_2, x_3, x_4, x_5, \dots] \xrightarrow{(1 \otimes p) \circ \psi} \mathbb{F}_2[\xi_1, \xi_2, \dots] \otimes \mathbb{F}_2[x_2, x_4, x_5, \dots]$$

takes each ring generator x_i to the ring generator of the corresponding degree, plus a polynomial in the lower-degree generators. This yields a formula for obtaining each generator on the right-hand side as the image of some polynomial on the left-hand side. Therefore the map is surjective. Both sides have the same dimension, so the map is an isomorphism of rings.

7 The Main Result

In the last section we showed that the quotient

$$H_*(MO) \cong \mathbb{F}_2[x_1, x_2, x_3, x_4, x_5, \dots] \rightarrow \mathbb{F}_2[x_2, x_4, x_5, \dots] = N_*$$

gave an isomorphism

$$\mathcal{A} \otimes N^* \rightarrow \mathcal{A} \otimes H^*(MO) \rightarrow H^*(MO)$$

Therefore $H^*(MO)$ is a free \mathcal{A} -module with basis given by the image of some basis for N^* as a vector space over \mathbb{F}_2 . Choose the basis dual to the monomials in N_* . Then each dual basis element in N^k pushes forward to an element of $H^k(MO) \cong \tilde{H}^{k+n}(MO(n))$ ($0 \leq k \leq n$) that is classified by a map into the Eilenberg-MacLane space $K(\mathbb{F}_2, k+n)$. From a previous section, $H^*(K(\mathbb{F}_2, k+n))$ is isomorphic to a free \mathcal{A} -module on the fundamental class up through degree $2(k+n) - 1$. Therefore the product of these classifying maps gives an isomorphism

$$H^*\left(\prod_i K(\mathbb{F}_2, n_i)\right) \rightarrow H^*(MO(n))$$

through degree $2n-1$. On the spectrum level, the classifying maps $MO(n) \rightarrow K(\mathbb{F}_2, n+k)$ become spectrum maps $MO \rightarrow \Sigma^{-k}\mathbb{H}\mathbb{F}_2$ that induce isomorphisms on \mathbb{F}_2 cohomology in all degrees. This is captured in the following diagram, which commutes by our construction.

$$\begin{array}{ccc}
 H^*(MO) & \xrightarrow{p} & N^* \\
 \uparrow \cong & & \uparrow \\
 H^*(\bigvee_i \Sigma^{-k_i} \mathbb{H}\mathbb{F}_2) & & \\
 \uparrow \cong & & \swarrow \\
 \mathcal{A} \otimes N^* & &
 \end{array}$$

Therefore, by the \mathcal{C}_2 -approximation theorem, the product map $MO \rightarrow \prod_i \Sigma^{-k_i} \mathbb{H}\mathbb{F}_2$ gives a homotopy equivalence modulo odd torsion. We already know that both spectra have no odd torsion, so it is a homotopy equivalence.

Consider the diagram

$$\begin{array}{ccccccc}
\mathcal{N} & \xrightarrow{\cong} & \pi_*(MO) & \xrightarrow{h} & H_*(MO) & \xrightarrow{p} & N_* \\
& & \downarrow \cong & & \downarrow \cong & & \uparrow \\
& & \pi_*(\bigvee_i \Sigma^{-k_i} \mathbb{H}\mathbb{F}_2) & \xrightarrow{h} & H_*(\bigvee_i \Sigma^{-k_i} \mathbb{H}\mathbb{F}_2) & & \\
& & \uparrow \cong & & \uparrow \cong & & \\
N^* & \xrightarrow{\quad} & \mathcal{A}^* \otimes N_* & & & &
\end{array}$$

where the vertical maps come from the homotopy equivalence just described, and the horizontal maps in the top row are the Pontryagin-Thom map from section 2, the Hurewicz homomorphism, and the quotient described above. The top square commutes by the naturality of the Hurewicz homomorphism. By construction, the bottom zigzag composes to give the isomorphism $N^* \rightarrow N_*$ taking the dual basis on N^* to the basis of monomials on N_* . Therefore it is an isomorphism. Therefore the top maps compose to give an isomorphism of groups. We already know that the first and last maps on the top are maps of rings. The middle map is a map of rings as well, which we see by taking the limit of

$$\begin{array}{ccc}
\pi_*(MO(k) \wedge MO(\ell)) & \xrightarrow{h} & \tilde{H}_*(MO(k) \wedge MO(\ell)) \\
\downarrow & & \downarrow \\
\pi_*(MO(k+\ell)) & \xrightarrow{h} & \tilde{H}_*(MO(k+\ell))
\end{array}$$

which commutes again by naturality of the Hurewicz homomorphism.

Therefore the top row of maps gives a ring isomorphism

$$\mathcal{N}_* \rightarrow N_* \cong \mathbb{F}_2[x_2, x_4, x_5, \dots]$$

which is the main result.

An easy corollary follows: the first two maps in the top row form an injective map of rings. Composing this with the Thom isomorphism $H_*(MO) \rightarrow H_*(BO)$ gives an injective ring map

$$\mathcal{N} \rightarrow H_*(BO) \cong \text{Hom}(H^*(BO), \mathbb{F}_2)$$

Tracing through the Pontryagin-Thom construction, we see that for a specific manifold M of dimension n , the image of $[M]$ in $\text{Hom}(H^n(BO), \mathbb{F}_2)$ takes each degree n monomial in the w_i to a number. This number is computed by taking the monomial in $w_i(\nu)$ and pairing it with the fundamental class of M . These numbers are called the *Stiefel-Whitney numbers*. The existence of this map from \mathcal{N} proves the easy fact that cobordant manifolds have the same Stiefel-Whitney numbers. The injectivity of the map proves the difficult fact that two manifolds with identical Stiefel-Whitney numbers are in fact cobordant.

References

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