Universal algebraic extensions

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Abstract

This note records an in-progress attempt to provide a rigorous answer to the question: "What does it mean to say that θ-algebras capture *all the structure* of the p-adic K-theory of an E_∞-ring?" We propose a categorical framework in which this notion is encoded as a universal property.

This attempt would not have reached its current stage without the helpful conversations we've had with Martin Frankland, Justin Noel, Herman Stel, and Karol Szumilo. Of course, any of the remaining faults (and there are sure to be many) are ours alone.

Contents

1 Definitions.

Let $L : C \leftrightarrows D : R$ be an adjunction of "topological" categories, and let $F : C \to A$ be a functor to an "algebraic" category. We define a new category $\texttt{AlgEx} = \texttt{AlgEx}_{F, L \dashv R}$ of *algebraic extensions* of F along $L \dashv R$ as follows.

- An **object** of AlgEx consists of a tuple $(X, G_X, L_X, R_X, \eta_X^{restr}, \eta_X^{enr})$. Here, $G_X : D \to X$ is a functor and $L_X : A \leftrightarrows X : R_X$ is an adjunction, and these fit into the diagram

$$\begin{array}{ccc} C & \underset{R}{\overset{L}{\leftrightarrows}} & D \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G_X} \\ A & \underset{R_X}{\overset{L_X}{\leftrightarrows}} & X. \end{array}$$

  The last two elements are:

  - a natural <u>isomorphism</u> $\eta_X^{restr} : R_X G_X \Rightarrow FR \in \texttt{Fun}(D, A)$, and
  - a natural transformation $\eta_X^{enr} : L_X F \Rightarrow G_X L \in \texttt{Fun}(C, X)$.

  The datum η_X^{restr} ensures that G_X *restricts* to the original functor F, and for $c \in C$ an RL-algebra the datum η_X^{enr} *enriches* $F(c)$ from simply an object of A to an $R_X L_X$-algebra[1]. Explicitly, if $c \in \texttt{Alg}_{RL}$ via $RL(c) \xrightarrow{\alpha} c$, then $F(c) \in \texttt{Alg}_{R_X L_X}$ via

$$R_X L_X(F(c)) \xrightarrow{R_X(\eta_X^{enr}(c))} R_X G_X L(c) \xrightarrow{\eta_X^{restr}(L(c))} FRL(c) \xrightarrow{F(\alpha)} F(c).$$

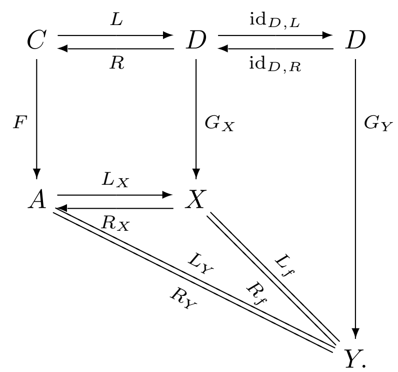
[1] Should we be restricting to the case of a monadic adjunction (i.e., $R : D \xrightarrow{\sim} \texttt{Alg}_{RL,C}$) for any reason?

Given a diagram of categories and adjunctions as above, there is a bijection

$$\mathrm{Hom}_{\mathtt{Fun}(D,A)}(FR, R_X G_X) \cong \mathrm{Hom}_{\mathtt{Fun}(C,X)}(L_X F, G_X L)$$

under which two corresponding natural transformations are said to be *mated*. We require that $(\eta_X^{restr})^{-1}$ and η^{enr} are mated. Of course, this means that we could equivalently specify just one of η_X^{restr} and η_X^{enr}; in particular, for expository reasons we might like to keep only η_X^{restr}, so that we are very clearly asking for nothing more than a factorization of FR through some potentially richer category. But already in the "transfer of algebra structure" formula we have seen that both are useful, so we stick to the present formulation. We justify this requirement in the section below entitled *The mates requirement.*

- A **morphism** $f : (X, G_X, L_X, R_X, \eta_X^{restr}, \eta_X^{enr}) \to (Y, G_Y, L_Y, R_Y, \eta_Y^{restr}, \eta_Y^{enr})$ in `AlgEx` consists of a tuple $(L_f, R_f, \lambda_f, \rho_f, \chi_f, \varepsilon_f)$. Here, $L_f : X \leftrightarrows Y : R_f$ is an adjunction, which fits into the diagram[2,3]



The last four elements are:

- a *left weakening* natural transformation $\lambda_f : L_f L_X \Rightarrow L_Y \in \mathtt{Fun}(A, Y)$,
- a *right weakening* natural transformation $\rho_f : R_X R_f \Rightarrow R_Y \in \mathtt{Fun}(Y, A)$,
- a *compatibility* natural isomorphism $\chi_f : R_f G_Y \Rightarrow G_X \mathrm{id}_{D,R} \in \mathtt{Fun}(D, X)$, and
- an *extension* natural transformation $\varepsilon_f : L_f G_X \Rightarrow G_Y \mathrm{id}_{D,L} \in \mathtt{Fun}(D, Y)$.

These natural transformations are required to satisfy:

- the *compatibility* condition that

$$\begin{array}{ccccc} R_X R_f G_Y & \xRightarrow{\ \mathrm{Id}_{R_X} \circ \chi_f\ } & & & R_X G_X \mathrm{id}_{D,R} \\ \Big\Downarrow {\scriptstyle \rho_f \circ \mathrm{Id}_{G_Y}} & & & & \Big\Downarrow {\scriptstyle \eta_X^{restr} \circ \mathrm{Id}_{\mathrm{id}_{D,R}}} \\ R_Y G_Y & \xRightarrow[\ \eta_Y^{restr}\ ]{} & FR & = & FR\mathrm{id}_{D,R} \end{array}$$

  commutes in $\mathtt{Fun}(D, A)$, and
- the *extension* condition that

$$\begin{array}{ccccc} L_f L_X F & \xRightarrow{\ \mathrm{Id}_{L_f} \circ \eta_X^{enr}\ } & & & L_f G_X L \\ \Big\Downarrow {\scriptstyle \lambda_f \circ \mathrm{Id}_F} & & & & \Big\Downarrow {\scriptstyle \varepsilon_f \circ \mathrm{Id}_L} \\ L_Y F & \xRightarrow[\ \eta_Y^{enr}\ ]{} & G_Y L & = & G_Y \mathrm{id}_{D,L} L \end{array}$$

  commutes in $\mathtt{Fun}(C, Y)$.

[2]We use two lines in this diagram since our diagrams package can't pile diagonal arrows. Of course this will be amended in a future version. By convention, left adjoints are labeled above and right adjoints are labeled below.

[3]We use $\mathrm{id}_{D,L}$ and $\mathrm{id}_{D,R}$ to keep things clear, but we draw the line before distinguishing between the two copies of D in the diagram.

We also require that $(\chi_f)^{-1}$ and ε_f are mated.

Note that such a morphism can therefore be thought of as precisely the data of:

- an object $(Y, G_Y, L_f, R_f, \chi_f, \varepsilon_f) \in \texttt{AlgEx}_{G_X, \mathrm{id}_{D,L} \dashv \mathrm{id}_{D,R}}$, along with
- a weakening of the composite adjunction $L_f L_X \dashv R_X R_f$ via λ_f and ρ_f to some possibly more relaxed adjunction $L_Y \dashv R_Y$, satisfying
- the conditions necessary to make the composite rectangle into an object of $\texttt{AlgEx}_{F, \mathrm{id}_{D,L} L \dashv R \mathrm{id}_{D,R}} = \texttt{AlgEx}_{F, L \dashv R}$.

In particular, we should think of Y as a richer category than X through which we are factoring FR. This motivates us to define a **universal algebraic extension** of F along $L \dashv R$ to be a terminal object of $\texttt{AlgEx}$.[4,5]

2 Examples.

Here are the examples we have in mind. The first is something of a testing ground since it is so simple, and we will revisit it in the section *The mates requirement* in some detail.

Example 1. Let $C = \texttt{AbGrp}$ and $D = \texttt{CRing}$, with the usual monadic adjunction $\mathrm{Sym}^* : \texttt{AbGrp} \leftrightarrows \texttt{CRing} : (-, +)$. Then we can consider the forgetful functor $U_+ : \texttt{AbGrp} \to \texttt{Set}$. We have the evident string of morphisms $\texttt{Set} \to \texttt{AbSemigrp} \to \texttt{AbMon}$ in $\texttt{AlgEx}$.[6] We expect that $\texttt{AbMon}$ is universal.

This example reflects the intuition that if an abelian group is secretly a commutative ring, then even if we forget the abelian group structure, we still have an abelian monoid.

Before we get to the real examples of interest, we note with some amusement that we can flip this first example around.

Example 2. Let $C = \texttt{AbMon}$ and $D = \texttt{CRing}$, with the usual monadic adjunction $\mathbb{Z}[-] : \texttt{AbMon} \leftrightarrows \texttt{CRing} : (-, \times)$. Then we can consider the forgetful functor $U_\times : \texttt{AbMon} \to \texttt{Set}$. We have the evident string of morphisms $\texttt{Set} \to \texttt{AbSemigrp} \to \texttt{AbMon} \to \texttt{AbGrp}$ in $\texttt{AlgEx}$. We expect that $\texttt{AbGrp}$ is universal.

Of course, this example reflects a "dual" intuition compared to the first example.

We now turn to some examples of a topological flavor.

Example 3. Let $C = D = \texttt{Top}_{*,0}$ be the category of pointed connected[7] spaces, and take $L = R = \mathrm{id}_{\texttt{Top}_{*,0}}$. Then we can consider $\pi_* : \texttt{Top}_{*,0} \to \texttt{GrSet}$ to be the *graded homotopy set* functor (in positive dimensions). We then have the evident string of morphisms $\texttt{GrSet} \to \texttt{GrGrp} \to \pi_1\texttt{-Mod} \to \pi_1\texttt{-Mod}^{\texttt{Wh}} \to \Pi\texttt{-Alg}$ in $\texttt{AlgEx}$ (where $\texttt{Wh}$ denotes that we add the structure of Whitehead products), and we conjecture that $\Pi\texttt{-Alg}$ is the universal algebraic extension.[8] If this is true, then it will probably follow from Yoneda's lemma, along with the following fact: if $\{n_\beta\}_{\beta \in B}$ is a set of nonnegative integers[9] and $F_\Pi : \texttt{GrSet} \to \Pi\texttt{-Alg}$ denotes the free Π-algebra functor, then $\pi_* \left(\bigvee_{\beta \in B} S^{n_\beta} \right) \cong F_\Pi \left(\{n_\beta\}_{\beta \in B} \right)$ as Π-algebras (where of course we consider $\{n_\beta\}_{\beta \in B}$ as a graded set by saying that at level n it consists of the set $\{\beta \in B : n_\beta = n\}$).

Example 4. Let $C = \texttt{Spectra}$ and $D = E_\infty\texttt{-Rings}$ with the usual operadic-monadic adjunction $\mathbb{P} : \texttt{Spectra} \leftrightarrows E_\infty\texttt{-Rings} : U$. Then we can consider $K_* : \texttt{Spectra} \to \texttt{MoravaMod}$, where K_* denotes p-adic K-theory. We have the algebraic extension $K_* : E_\infty\texttt{-Rings} \to \theta\texttt{-Alg}$, and we've seen it implied that this is universal. For

[4] Note that $(A, FR, \mathrm{id}_A, \mathrm{id}_A, \mathrm{Id}_{FR}, F(\eta))$ is an initial object, where $\eta : \mathrm{id}_C \Rightarrow RL$ is the unit of the adjunction; one might therefore call this a "trivial" algebraic extension.

[5] In our examples, our algebraic adjunctions will indeed be monadic. This implies, for instance, that the right adjoint is *conservative* (i.e. it creates isomorphisms) and that it takes reflexive coequalizers to split coequalizers. Thus our right adjoints really are simply forgetting structure, and so it really does make sense for us to consider a terminal object universal.

[6] Recall that an *abelian semigroup* is a set equipped with a commutative associative binary operation, and an *abelian monoid* is an abelian semigroup which has an identity element. We forbear the inclusion of such categories as "magmas", which are primarily of interest these days as first examples of algebras over an operad.

[7] Possibly one could generalize from Π-algebras to Π-*algebroids*.

[8] Recall that a Π-*algebra* is a $\texttt{Set}$-valued product-preserving functor off the opposite of the category Π of arbitrary wedges of positive-dimensional spheres and based homotopy classes of maps.

[9] or more precisely, a function $B \to \mathbb{N}_0$ – of course we may have $n_\beta = n_{\beta'}$ for $\beta \neq \beta'$

instance, in Goerss-Hopkins's *Moduli Problems for Structured Ring Spectra*, they make this claim prosaically as a lead-up to their theorem 2.2.11. In our terminology and notation, that theorem says that this is indeed an algebraic extension, and moreover that if $c \in \mathtt{Spectra}$ is cofibrant and $K_* c \in \mathtt{MoravaMod}$ is torsion free, then $\eta^{enr}_{\theta\text{-}\mathtt{Alg}}(c) : L_{\theta\text{-}\mathtt{Alg}} K_*(c) \to K_* \mathbb{P}(c)$ is an isomorphism.[10,11]

Example 5. Let E be any (not necessarily structured) ring spectrum such that E_*E is a flat E_*-module. Then we can take $C = D = \mathtt{Spectra}$, $L = R = \mathrm{id}_{\mathtt{Spectra}}$, and we can consider the functor $E_* : \mathtt{Spectra} \to \mathtt{Mod}_{E_*}$. We have the algebraic extension $E_* : \mathtt{Spectra} \to \mathtt{Comod}_{E_*E}$, and this is probably universal too.

3 The Yoneda follies: why this project isn't trivial.

There is one very important word which has not entered into the main discussion yet: *Yoneda.* Namely, suppose we have our generic situation $L : C \leftrightarrows D : R$ and $F : C \to A$. To determine all the structure that the target of F carries (i.e. to try to understand $\mathtt{AlgEx}_{F, \mathrm{id}_{C,L} \dashv \mathrm{id}_{C,R}}$), one might pass to the category $\mathtt{Presh}_A(C)$ of A-valued presheaves on C. Assuming A is tensored over $\mathtt{Set}$ (i.e. assuming it has a terminal object), C embeds into $\mathtt{Presh}_A(C)$ via the covariant Yoneda functor; in this way, we can consider F as being on an equal footing with the objects of C. Then it is tempting to declare that by Yoneda's lemma, all the possible structure on the functor represented by F is encoded in $\mathrm{End}_{\mathtt{Presh}_A(C)}(F)$. Going further, one might also declare that all the possible structure on the image of the pullback of F along $\iota : \mathtt{Alg}_{RL} \to C$ is encoded in $\mathrm{End}_{\mathtt{Presh}_A(\mathtt{Alg}_{RL})}(\iota^* F)$.

However, there are three reasons why we are not satisfied by this perspective.

- First of all, while this might seem to answer the question, as far as we are aware it gives virtually no concrete understanding whatsoever: in a field so propelled by down-to-earth computations as algebraic topology, this should not be considered a computable solution.

- The second reason is somewhat more damning: $\mathrm{End}(F)$ doesn't carry all the structure on the image of F. Let us return to the example of Π-algebras. The functor $\pi_* : \mathtt{Top}_{*,0} \to \mathtt{GrSet}$ is corepresented in $Ho(\mathtt{Top}_{*,0})$ at level n by the object S^n. Thus, $\mathrm{End}(\pi_*) \cong \prod_{n \geq 1} [S^n, S^n]_*$. We can recognize this product of hom-sets as a product of monoids (under composition) isomorphic to $\mathbb{Z}$. On the other hand, if we happen to remember that there are interesting degree-shifting transformations, then (up to determining a suitable method of bookkeeping) we obtain an action of all the (unstable) homotopy groups of spheres. But by the failure of excision for homotopy groups – that is, because in general $[S^m, \bigvee_{\beta \in B} S^{n_\beta}]_* \not\cong \prod_{\beta \in B} [S^m, S^{n_\beta}]_*$ – there is strictly more information encoded in a Π-algebra.[12] Of course, this might suggest that in general, we should simply study $\mathrm{Hom}_{\mathtt{Fun}(C^m, A)}(F^m, F)$ for all m. But the fact remains that our initial reliance on Yoneda's lemma led us astray, and there is no a priori reason (as far as we can see, at least) that perhaps we're not still being dense and there's some even further refinement of which we're simply not aware.

- But it is the third reason that truly drives the nail into the coffin, we think: $\mathrm{End}(F)$ *really* doesn't carry all the structure on the image of F! Suppose we are still looking at Π-algebras, but we decide to be clever – positive-dimensional homotopy sets *are* groups, after all – and begin with the functor $\pi_* : \mathtt{Top}_{*,0} \to \mathtt{GrGrp}$. Then we would completely miss most of the Π-algebra operations: unless a morphism in Π happens to be a map of co-H-spaces, then it will not induce a group homomorphism. Note that in our definition of a morphism in $\mathtt{AlgEx}$, the presence of the weakening natural transformations λ_f and ρ_f allows for the existence a morphism $\mathtt{GrGrp} \to \Pi\text{-}\mathtt{Alg}$.

  This might seem artificial, but consider the case that A is no longer a concrete category. Then it would be impossible to know[13] whether A is *sufficiently initial* that we can extend forward as far as is truly possible: we might be able to "extend backwards in order to extend forwards". We immediately acknowledge that our setup currently suffers from this deficiency too; however, in a future revision we plan to expand on the

[10] Of course, this reminds us that we should expand our theory to include the case that $L \dashv R$ is a Quillen adjunction. Perhaps not much will need to be changed.

[11] This suggests a possible general criterion for universality, namely that $\eta^{enr}_X(c)$ is an isomorphism for some suitably large class of objects $c \in C$.

[12] We note here that since spheres are $\aleph_0$-compact objects in $Ho(\mathtt{Top}_{*,0})$, a Π-algebra is equivalent to an algebra (in sets) over the symmetric colored operad whose colors are given by the natural numbers and whose set of multimorphisms $(n_1, \ldots, n_k) \to m$ is given by $[S^m, \bigvee_{1 \leq i \leq k} S^{n_i}]_*$.

[13] as far as our own limited knowledge of category theory takes us, at least

observation that morphisms are also objects in a different category of algebraic extensions, so that we can recognize our current framework as a *relative* version of a more general *absolute* theory.

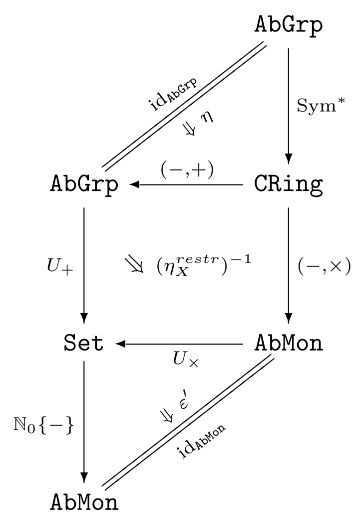
We hope that these reasons convince the reader that the motivating question for this project is not adequately resolved by Yoneda's lemma.

4 The mates requirement.

In this section, we justify the requirements in the definition of $\mathtt{AlgEx}$ that various pairs of natural transformations be mated. We do this through the use of an illuminating example.

Let us write $\mathrm{Sym}^* : \mathtt{AbGrp} \leftrightarrows \mathtt{CRing} : (-,+)$ for the usual monadic adjunction. We would like to study algebraic extensions of the functor $U_+ : \mathtt{AbGrp} \to \mathtt{Set}$. In particular, we study the algebraic extension $(X, G_X, L_X, R_X, \eta_X^{restr}, \eta_X^{enr}) = (\mathtt{AbMon}, (-,\times), \mathbb{N}_0\{-\}, U_\times, \eta_X^{restr}, \eta_X^{enr})$, where $\eta_X^{restr} : U_\times(-,\times) \Rightarrow U_+(-,+)$ is the evident natural isomorphism and $\eta_X^{enr} : \mathbb{N}_0\{U_+(-)\} \Rightarrow (\mathrm{Sym}^*(-), \times)$ is the natural transformation which acts as the identity function on generators and extends using the monoid structures.

To make our point, it will be useful for us to explicitly check that this is an object of $\mathtt{AlgEx}$, i.e. that $(\eta_X^{restr})^{-1}$ and η_X^{enr} are indeed mated. The mating bijection takes $(\eta_X^{restr})^{-1}$ to



(where η denotes the unit of $\mathrm{Sym}^* \dashv (-,+)$ and ε' denotes counit of $\mathbb{N}_0\{-\} \dashv U_\times$), considered as a natural transformation in

$$\mathrm{Hom}_{\mathtt{Fun(AbGrp,AbMon)}}(\mathbb{N}_0\{U_+(-)\}, (\mathrm{Sym}^*(-), \times)).$$

To see what this natural transformation is, let A be an arbitrary abelian group. Then we carry out the vertical compositions

$$\begin{array}{cl}
\mathbb{N}_0\{-\} \circ U_+ \circ \mathrm{id}_{\mathtt{AbGrp}} & \\
\Downarrow & \mathrm{Id}_{\mathbb{N}_0\{-\}} \circ \mathrm{Id}_{U_+} \circ \eta \\
\mathbb{N}_0\{-\} \circ U_+ \circ (-,+) \circ \mathrm{Sym}^* & \\
\Downarrow & \mathrm{Id}_{\mathbb{N}_0\{-\}} \circ (\eta_X^{restr})^{-1} \circ \mathrm{Id}_{\mathrm{Sym}^*} \\
\mathbb{N}_0\{-\} \circ U_\times \circ (-,\times) \circ \mathrm{Sym}^* & \\
\Downarrow & \varepsilon' \circ \mathrm{Id}_{(-,\times)} \circ \mathrm{Id}_{\mathrm{Sym}^*} \\
\mathrm{id}_{\mathtt{AbMon}} \circ (-,\times) \circ \mathrm{Sym}^* &
\end{array}$$

on A, using underline for simplicity to denote underlying set, as

$$
\begin{array}{ccccccccc}
\mathbb{N}_0\{\underline{A}\} & \xleftarrow{\mathbb{N}_0\{-\}} & \underline{A} & \xleftarrow{U_+} & A & & \xleftarrow{\mathrm{id}_{\texttt{AbGrp}}} & & A \\
\downarrow{\scriptstyle n\cdot\underline{a}\mapsto n\cdot\underline{1\cdot a}} & \mathrm{Id}_{\mathbb{N}_0\{-\}} & \downarrow{\scriptstyle \underline{a}\mapsto\underline{1\cdot a}} & \mathrm{Id}_{U_+} & \downarrow{\scriptstyle a\mapsto 1\cdot a} & & \eta & & \| \\
\mathbb{N}_0\{\underline{\mathrm{Sym}^*(A)}\} & \xleftarrow{\mathbb{N}_0\{-\}} & \underline{\mathrm{Sym}^*(A)} & \xleftarrow{U_+} & (\mathrm{Sym}^*(A),+) & \xleftarrow{(-,+)} & \mathrm{Sym}^*(A) & \xleftarrow{\mathrm{Sym}^*} & A \\
\downarrow{\scriptstyle \mathrm{id}} & \mathrm{Id}_{\mathbb{N}_0\{-\}} & \downarrow{\scriptstyle \mathrm{id}} & & (\eta_X^{restr})^{-1} & & \downarrow{\scriptstyle \mathrm{id}} & \mathrm{Id}_{\mathrm{Sym}^*} & \| \\
\mathbb{N}_0\{\underline{\mathrm{Sym}^*(A)}\} & \xleftarrow{\mathbb{N}_0\{-\}} & \underline{\mathrm{Sym}^*(A)} & \xleftarrow{U_\times} & (\mathrm{Sym}^*(A),\times) & \xleftarrow{(-,\times)} & \mathrm{Sym}^*(A) & \xleftarrow{\mathrm{Sym}^*} & A \\
\downarrow{\scriptstyle n\cdot\underline{f(a_i)}\mapsto f(a_i^n)} & & \varepsilon' & & \downarrow{\scriptstyle \mathrm{id}} & \mathrm{Id}_{(-,\times)} & \downarrow{\scriptstyle \mathrm{id}} & \mathrm{Id}_{\mathrm{Sym}^*} & \| \\
(\mathrm{Sym}^*(A),\times) & & \xleftarrow{\mathrm{id}_{\texttt{AbMon}}} & & (\mathrm{Sym}^*(A),\times) & \xleftarrow{(-,\times)} & \mathrm{Sym}^*(A) & \xleftarrow{\mathrm{Sym}^*} & A.
\end{array}
$$

(Here, $f(a_i) \in \underline{\mathrm{Sym}^*(A)}$ denotes an arbitrary polynomial.) Therefore, our new natural transformations evaluated on A yields the morphism in $\mathrm{Hom}_{\texttt{AbMon}}(\mathbb{N}_0\{\underline{A}\}, (\mathrm{Sym}^*(A), \times)$ given by

$$n \cdot \underline{a} \mapsto n \cdot \underline{1 \cdot a} \mapsto n \cdot \underline{1 \cdot a} \mapsto 1 \cdot a^n.$$

This is indeed η_X^{enr}.

Now, we can finally explain the origin of the mates requirements. Let us temporarily write $\texttt{AlgEx}^?$ for the category whose objects and morphisms are all the same as those of $\texttt{AlgEx}$ but without the mates requirements. Then we can define an object $Y^{k?} \in \texttt{AlgEx}^?_{U_+,\mathrm{Sym}^*\dashv(-,+)}$ to be exactly the same as the object $X \in \texttt{AlgEx}_{U_+,\mathrm{Sym}^*\dashv(-,+)}$ above, except that we define $\eta^{enr}_{Y^{k?}}$ by $n \cdot \underline{a} \mapsto 1 \cdot a^{kn}$ for some fixed $k \geq 1$. Then X and $Y^{k?}$ determine precisely the same factorizations of $U_+(-,+) : \texttt{CRing} \to \texttt{Set}$, but there is an obvious (nontrivial and noninvertible for $k > 1$) morphism $X \to Y^{k?}$ in $\texttt{AlgEx}^?$; one can easily check as we have done above that $\eta^{enr}_{Y^{k?}}$ is mated to the natural transformation in $\mathrm{Hom}_{\texttt{Fun(CRing,Set)}}(U_+(-,+), U_\times(-,\times))$ given by $\underline{r} \mapsto \underline{r^k}$. Thus, $X = Y^{1?}$ is initial among the $Y^{k?}$. This should be a more general phenomenon: if we are given a fixed natural isomorphism η^{restr}, then among all the η^{enr} that we might pair it with in $\texttt{AlgEx}^?$, its mate is initial.

Thus, the mates requirement for objects of $\texttt{AlgEx}$ allows for our category to have a terminal object, as we would hope.[14] Otherwise, more or less all of our objects will admit nontrivial endomorphisms, and our desired universal algebraic extension will only have the much weaker universal property that it admits a morphism from every other object. (And of course, once we have made this decision for our objects, clearly we must have it for our morphisms as well.)

[14] We see a vague analogy $\texttt{AlgEx}^?$: $\texttt{AlgEx}$ :: suspension spectra : Ω-spectra, since in both pairs, the latter category has a requirement that certain morphisms be adjoint to equivalences. We note that there might be a possibility of something like a model structure on $\texttt{AlgEx}^?$, where $\texttt{AlgEx}$ is the (probably full) subcategory of cofibrant objects; note that applying such a replacement on the source would give an inverse to $X \to Y^{k?}$ when we take derived maps. But we haven't looked into this very much. Note that the axioms for a model category imply the existence of a terminal object, so a priori it'll be at least as hard to prove we have a model category as it will be to show that universal algebraic extensions exist.