

Stratifications

in Algebra & Topology

§1: Recollements

§2: Stratifications

§3: Θ -monoidal stratifications

e.g. monoidal, braided monoidal, symmetric monoidal

§4: Equivariant cohomology

joint with David Ayala & Nick Rozenblyum

v1: arXiv 1910.14602

v1.9: etale.site/writing/strat.pdf

these slides: etale.site/writing/msri-strat.pdf

Conventions: category := presentable stable ∞ -category ; scheme := nice scheme

quasicompact, noetherian, finite-dimensional

e.g. pretriangulated dg-cat
with infinite sums

§1: Recollements

Prototype: Reconstruct $M \in \text{Mod}_{\mathbb{Z}}$ from

↪ p -completions M_p^\wedge ,

↪ rationalization $\mathbb{Q} \otimes_{\mathbb{Z}} M$,

↪ gluing data.

$$\begin{array}{ccc} M & \xrightarrow{\quad} & \mathbb{Q} \otimes_{\mathbb{Z}} M \\ \downarrow & \lrcorner & \downarrow \text{gluing map} \\ \varprojlim_{p \text{ prime}} M_p^\wedge & \xrightarrow{\quad} & \mathbb{Q} \otimes_{\mathbb{Z}} \left(\varprojlim_{p \text{ prime}} M_p^\wedge \right) \end{array}$$

ALGEBRA

TOPOLOGY

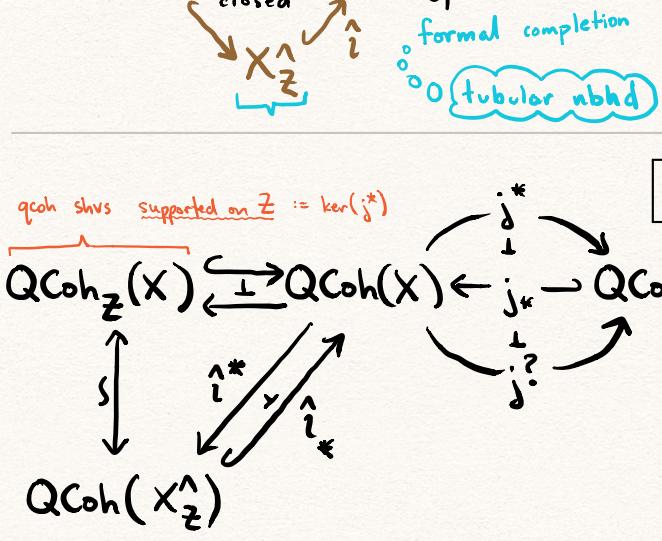
closed-open decomposition

Schemes

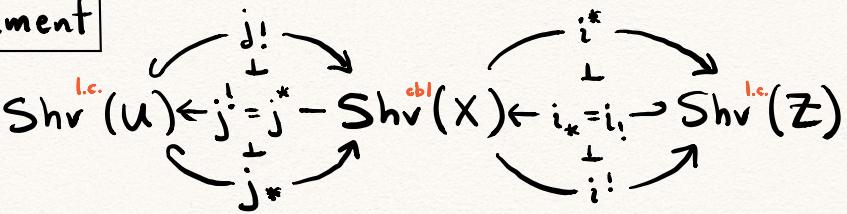
$$\mathbb{Z} \xrightarrow{i \text{ closed}} X \xleftarrow{j \text{ open}} U$$

topological spaces

$$U \xleftarrow{j} X \xleftarrow{i} \mathbb{Z}$$



recollement



l.c. := locally constant

$\text{cbl} := \text{constructible}$ (w.r.t. this closed open decomp.)

$$\text{Spec}\left(k \xleftarrow{\circlearrowleft x} k[x] \longrightarrow k[x^\pm]\right)$$

doesn't preserve heart!

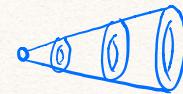
$$K \in Top \iff K \times (0,1] \hookrightarrow C(K) \xleftarrow{f_K} pt$$

$$K = S^1 :$$



10

$$K = T^2;$$



A blue line drawing showing three circles connected by a single horizontal line. The first circle is at the left end of the line. The second circle is positioned higher up and to the right. The third circle is positioned even higher up and to the right than the second one.

1

Diagram illustrating the relationship between Shv^{l.c.}(K), Shv^{cbl}(C(K)), and Shv^{l.c.}(pt) via functors Fun(Exit(C(K)), V).

```

    graph LR
      A[Shvl.c.(K)] <--> B[Shvcbl(C(K))]
      B <--> C[Shvl.c.(pt)]
      A -- "j*" --> B
      B -- "i*" --> C
      B -- "Fun(Exit(C(K)), V)" --> D[Fun(Pi_0(K)^d, V)]
      C -- "i*" --> D
      C -- "!" --> E[!stalk]
      E -- "i*" --> D
  
```

The diagram shows three main objects arranged horizontally: $\text{Shv}^{\text{l.c.}}(K)$, $\text{Shv}^{\text{cbl}}(C(K))$, and $\text{Shv}^{\text{l.c.}}(\text{pt})$. There are double-headed arrows between $\text{Shv}^{\text{l.c.}}(K)$ and $\text{Shv}^{\text{cbl}}(C(K))$, and between $\text{Shv}^{\text{cbl}}(C(K))$ and $\text{Shv}^{\text{l.c.}}(\text{pt})$. Below $\text{Shv}^{\text{l.c.}}(K)$ is $\text{Shv}^{\text{l.c.}}(K)$ with a circled '12'. Below $\text{Shv}^{\text{cbl}}(C(K))$ is $\text{Shv}^{\text{cbl}}(C(K))$ with a circled '12'. Below $\text{Shv}^{\text{l.c.}}(\text{pt})$ is $\text{Shv}^{\text{l.c.}}(\text{pt})$ with a circled '12' and a checkmark. Between $\text{Shv}^{\text{l.c.}}(K)$ and $\text{Shv}^{\text{cbl}}(C(K))$ is a downward arrow labeled j^* . Between $\text{Shv}^{\text{cbl}}(C(K))$ and $\text{Shv}^{\text{l.c.}}(\text{pt})$ are two arrows: an upward arrow labeled i^* and a downward arrow labeled i^* . Below the middle row is the functor $\text{Fun}(\text{Exit}(C(K)), V)$ with a circled '12'. Below $\text{Shv}^{\text{l.c.}}(\text{pt})$ is another copy of $\text{Shv}^{\text{l.c.}}(\text{pt})$ with a circled '12' and a checkmark. Between the bottom row and the functor are two arrows: an upward arrow labeled i^* and a downward arrow labeled $!$ -stalk.

Abstract Def" : A recollement is a diagram
s.t. all three sequences are exact.

$$\text{im}(i_L) = \ker(p_L), \quad \text{im}(\nu) = \ker(y), \quad \text{im}(i_R) = \ker(p_R)$$

QCoh notation

$y = \text{restricted Yoneda}$

$v = \text{null objects} \equiv \text{w.r.t. } y$

Microcosm Reconstruction Theorem: $\forall F \in X$, writing $F_0 = yF$ and $F_1 = p_L F$,

can reconstruct \tilde{F} as a pullback:

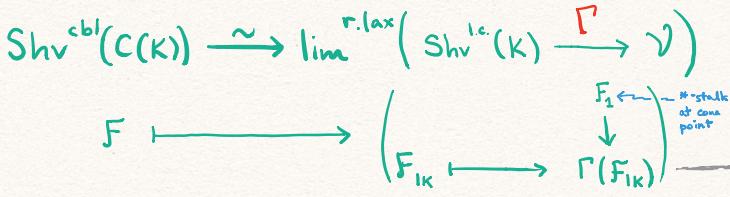
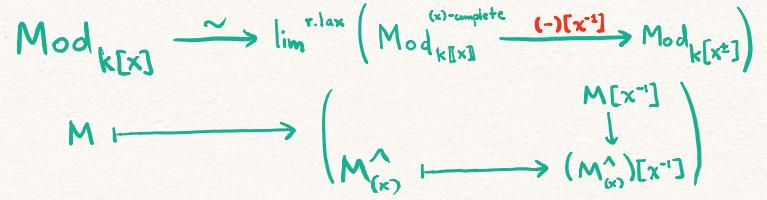
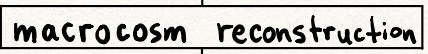
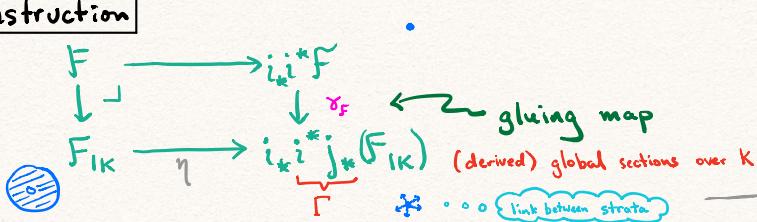
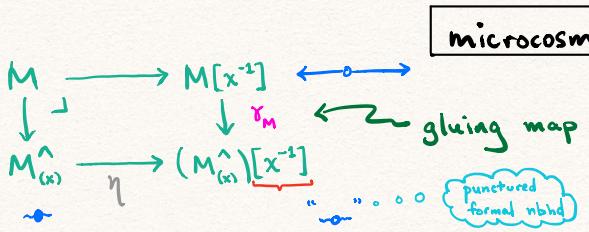
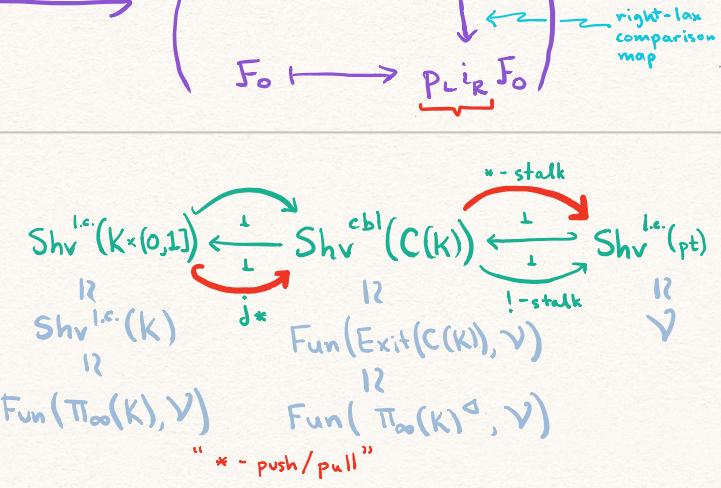
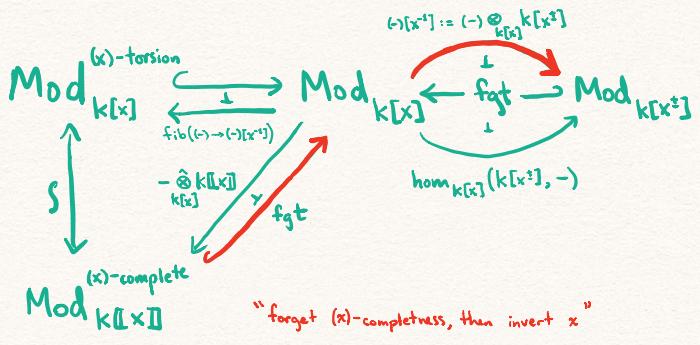
$$\begin{array}{ccc} F & \xrightarrow{\quad} & vF_1 \\ \downarrow & \lrcorner & \downarrow \\ i_R F_0 & \xrightarrow[\eta]{} & vPLi_R F_0 \end{array} \quad .$$

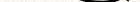
gluing map

...and this pullback square for F is unique:

Macrocosm Reconstruction Theorem:

$$x \xrightarrow{\sim} \lim^{\text{r.lax}}(z \xrightarrow{\text{PL ir}} u).$$



Rmk.: Verdier duality \Rightarrow reconstruct via  (e.g. ! - push/pull for coh shvs).

§2: Stratifications

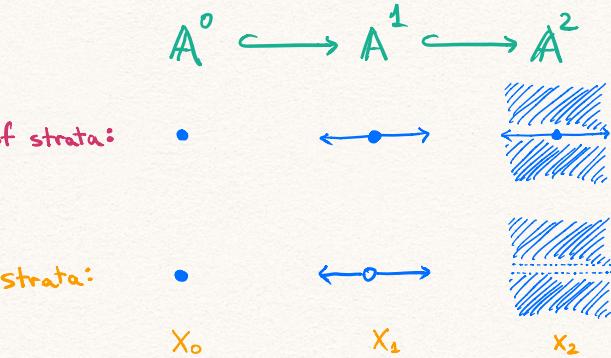
Defⁿ: A stratification of a scheme X over the poset $[2] = \{0 < 1 < 2\}$ is a functor $[2] \xrightarrow{\exists_0} \text{Cls}_X$ s.t. $\exists_2 = X$.

\Updownarrow closed subsets of X

$\exists_0 \subset \exists_1 \subset \exists_2 = X$ closed

closures of strata: • ↔•↔

$$\text{i}^{\text{th}} \text{ stratum: } X_i := Z_i \setminus Z_{i-1} \xrightarrow[\text{locally closed}]{} X$$



Defⁿ: A closed subcategory of \mathcal{X} is a full presentable stable subcat. $\mathcal{Z} \subseteq \mathcal{X}$ s.t.
 Qcoh terminology \exists adjoints $\mathcal{Z} \rightleftarrows_{F\dashv^{-1}} \mathcal{X}$.

$$\underline{\text{Ex.}}: \quad \mathcal{Z} \xrightarrow[\text{closed}]{} X \quad \rightsquigarrow \quad \mathbf{QCoh}_{\mathcal{Z}}(X) \xrightarrow[\text{closed}]{} \mathbf{QCoh}(X)$$

Def: A stratification of X over $[2]$ is a functor $[2] \xrightarrow{\cong} \mathbf{Cl}_X$ s.t. $\mathbb{Z}_2 = X$.

$$\text{Ex.: } [2] \xrightarrow{z_0} \text{Cls}_X \xrightarrow{\text{Qcoh}_{(z)}(X)} \text{Cls}_{\text{Qcoh}(X)}$$

closed subcats of X

$$i^{\text{th}} \text{ stratum: } X_i := Z_i / Z_{i-1} \quad \text{Ex.: } \text{Qcoh}(X)_i \simeq \text{Qcoh}((U_i) \widehat{\wedge} X_i)$$

$$X \xleftarrow[\rho^i]{\perp} X_i \rightsquigarrow \text{gluing functors: } \Gamma_i^j : X_i \xrightarrow{\rho^i} X \xrightarrow{\rho^j} X_j \quad \text{e.g. } \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

X_i locally closed $\rightarrow X$
closed $\rightarrow U_i$ open

~~~ gluing diagram:

$$\begin{array}{ccc} & X_1 & \\ \Gamma_0^1 & \uparrow & \Gamma_1^2 \\ X_0 & \xrightarrow{\Gamma_0^2} & X_2 \end{array}$$

$$\text{i.e. } [2] \xrightarrow[1.\text{lax}]{\perp} \text{Cat} \quad \text{a left-lax functor}$$

Ex.:

$$\begin{array}{ccc} Z_0 & \hookrightarrow & Z_1 & \hookrightarrow & Z_2 = X \\ \parallel & & \parallel & & \parallel \end{array}$$

$$\begin{array}{ccc} \text{Qcoh}((X \setminus Z) \widehat{\wedge} (Y \setminus Z)) & \xrightarrow{\quad} & \text{Qcoh}(X \widehat{\wedge} Z) \\ \uparrow & & \downarrow \\ \text{Qcoh}(X \widehat{\wedge} Y) & \xrightarrow{\quad} & \text{Qcoh}(X \setminus Y) \end{array}$$



vs.

### Macrocosm Reconstruction Theorem:

$$X \xrightarrow[\text{regulating}]{\sim} \lim^{r.\text{lax}} \left( \begin{array}{ccc} & X_1 & \\ \Gamma_0^1 & \uparrow \gamma & \Gamma_1^2 \\ X_0 & \xrightarrow{\Gamma_0^2} & X_2 \end{array} \right) := \left\{ \left( \begin{array}{c} F_0, \\ \vdots \\ F_n \end{array} \right), \quad \begin{array}{c} F_1 \\ \downarrow \tau_{01} \\ F_0 \\ \vdots \\ F_2 \\ \downarrow \tau_{12} \\ F_1 \\ \downarrow \tau_{02} \\ F_0 \end{array}, \quad \begin{array}{c} \text{right-lax limit of} \\ \text{left-lax functor} \\ \Gamma_1^2 F_0 \\ \circ \dots \circ \\ \Gamma_1^2 F_1 \\ \downarrow \eta \\ \Gamma_1^2 \Gamma_0^2 F_0 \end{array} \right\}$$

### Microcosm Reconstruction

$$X \ni \lim \left( \begin{array}{ccccc} "2" & \longrightarrow & "02" & & \\ \downarrow & & \downarrow & & \\ "0" & \longrightarrow & "01" & \longrightarrow & "012" \end{array} \right) \quad \text{"include back into } X \text{ via } X_i \xrightarrow{\rho^i} X\text{"}$$

Def<sup>n</sup>: A stratification of a scheme  $X$  over a poset  $P$  is a functor  $P \xrightarrow{z_0} \text{Cls}_X$  s.t.

$$\textcircled{1} \quad \bigcup_{p \in P} Z_p = X, \quad \text{generation condition}$$

$$\textcircled{2} \quad \forall p, q \in P, \quad Z_p \cap Z_q = \bigcup_{r \leq p \text{ & } r \leq q} Z_r. \quad \text{stratification condition}$$

need this!  
 $\{(0,0)\} \hookrightarrow \{y\text{-axis}\}$   
 $\{y\text{-axis}\} \hookrightarrow \mathbb{A}^2$

Def<sup>n</sup>: A stratification of a category  $X$  over a poset  $P$  is a functor  $P \xrightarrow{z_0} \text{Cls}_X$  s.t.

$$\textcircled{1} \quad \bigcup_{p \in P} Z_p = X \quad \text{generation condition}$$

$$\textcircled{2} \quad \forall p, q \in P, \quad \bigcup_{r \leq p \text{ & } r \leq q} Z_r \hookrightarrow Z_p \quad \text{stratification condition}$$

$$\text{Ex.: } P \xrightarrow{z_0} \text{Cls}_X \xrightarrow{\text{Qcoh}_{(z)}(X)} \text{Cls}_{\text{Qcoh}(X)}$$

$$\text{Ex.: } X \text{ a } P\text{-stratified top. spc.} \rightsquigarrow P^{\text{op}} \xrightarrow{U_0} \overline{\text{Open}}_X \xrightarrow{\text{Shv}} \overline{\text{Cls}}_{\text{Shv}(X)} \quad \text{?}$$

$$p^{\#} \text{ stratum: } X_p := Z_p / Z_{\leq p} \rightsquigarrow \dots$$

... ~~ macrocosm gluing diagram:  $G(X) : P \xrightarrow[1.\text{lax}]{\perp} \text{Cat}$

$$\begin{array}{ccc} p < q < r & \rightsquigarrow & \begin{array}{ccc} & X_r & \\ \uparrow & & \downarrow \\ X_p & \longrightarrow & X_r \end{array} \end{array}$$

$$p \mapsto X_p$$

a left-lax functor

$$\text{microcosm gluing diagram functor } X \xrightarrow{\quad g \quad} \lim_{\text{r.lax}} (\mathcal{G}_j(X))$$

$sd(P)$ : the subdivision of  $P$

$$sd([1]) = \begin{pmatrix} \rightarrow \\ \downarrow \end{pmatrix}, \quad sd([2]) = \begin{pmatrix} \rightarrow & \rightarrow \\ \downarrow & \rightarrow & \rightarrow \end{pmatrix}$$

5

$$\text{e.g. } F \longmapsto \begin{pmatrix} F_1 \\ F_0 \longmapsto p_{CIR} F_0 \end{pmatrix}$$

$$\underline{\text{Ex.}}: \chi = \text{Mod}_{\mathbb{Z}}, \quad P_{\mathbb{Z}} = \left( \begin{array}{ccccccc} & & (0) & & & & \\ & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ (2) & & (3) & & (5) & \dots & \end{array} \right) \xrightarrow{\quad \mathcal{Z}_0 \quad} \text{Cl}_{\text{Mod}_{\mathbb{Z}}} \\ \text{Mod}_{\mathbb{Z}}^{(2)-\text{torsion}} \quad \text{Mod}_{\mathbb{Z}}^{(3)-\text{torsion}} \dots$$

→ macrocosm gluing diagram

$$G(\text{Mod}_{\mathbb{Z}}) : P_{\mathbb{Z}} \longrightarrow \text{Cat}$$

depth 1  $\Rightarrow$  no laxness

$$G(\text{Mod}_{\mathbb{Z}}) = \left( \begin{array}{cccc} & \xrightarrow{\text{Mod}_{\mathbb{Q}}} & & \\ \xrightarrow{\mathbb{Q} \otimes_{\mathbb{Z}} (-)} & & \xleftarrow{\mathbb{Q} \otimes_{\mathbb{Z}} (-)} & \\ & \text{Mod}_{\mathbb{Z}_2^{\wedge}}^{(2)\text{-complete}} & \text{Mod}_{\mathbb{Z}_3^{\wedge}}^{(3)\text{-complete}} & \dots \end{array} \right)$$

$$\text{Mod}_{\mathbb{Z}} \xleftarrow{\lim_{sd(P_{\mathbb{Z}})}} \xrightarrow{\begin{matrix} g \\ \perp \end{matrix}} \lim_{P_{\mathbb{Z}}}^{\text{relax}} (G(\text{Mod}_{\mathbb{Z}}))$$

$$\lim \left( \prod_p M_p \rightarrow \prod_p (\mathbb{Q} \otimes_{\mathbb{Z}} M_p) \right) \quad \leftarrow \quad \begin{array}{c} M_0 \\ \downarrow \\ Q \otimes_{\mathbb{Z}} M_2 \\ \uparrow \eta \\ M_2 \end{array} \quad \begin{array}{c} \tau_2 \\ \downarrow \\ Q \otimes_{\mathbb{Z}} M_3 \\ \uparrow \eta \\ M_3 \end{array} \quad \begin{array}{c} \tau_3 \\ \downarrow \\ Q \otimes_{\mathbb{Z}} M_5 \\ \uparrow \eta \\ M_5 \end{array} \dots$$

## PROBLEM:

**PROBLEM:**

$$\lim \left( \prod_p \hat{M_P} \longrightarrow \underbrace{\mathbb{Q} \otimes_{\mathbb{Z}} (\prod_p \hat{M_P})}_{\text{arithmetic fracture}} \right) \simeq M \xrightarrow{\eta} \lim \left( \prod_p \hat{M_P} \longrightarrow \underbrace{\prod_p (\mathbb{Q} \otimes_{\mathbb{Z}} \hat{M_P})}_{\text{unit of adjunction}} \right)$$

**NOT AN EQUIVALENCE!**  
e.g. for  $M = \mathbb{Z}/2 \oplus \mathbb{Z}/3 \oplus \dots$

**THEOREM** (the *cosms*): Let  $P$  be a poset.

↳ metacosm adjunction Strat<sub>P</sub>  $\begin{array}{c} \xrightarrow{\text{G}} \\ \perp \\ \xleftarrow{\lim_{\text{r.lax}}} \end{array}$  LMod<sub>P</sub><sup>r.lax (1)</sup>

P-stratified cats

↳ unit at  $X \in \text{Strat}_P$  is macrocosm adjunction  $X \xleftarrow{\perp} \lim_{\substack{\rightarrow \\ \text{sd}(P)}}^{\text{r.lax}} (\mathcal{G}(X))$ ; (2)

↪ unit at  $F \in \mathcal{X}$  is microcosm morphism  $F \longrightarrow \lim_{sd(P)} (g(F))^{(3)}$  in  $\mathcal{X}$ ;

↪ ∀  $E \in X$ , get nano-cosm morphism  $\hom_X(E, F) \xrightarrow{\circlearrowleft} \lim_{\varphi \in \text{sd}(P)} (\hom_X(\Phi^{\varphi(n)} E, \Gamma^{\varphi(n)} \Phi^{\varphi(m)} F))$  (4) in  $\mathbf{Sp}$ .

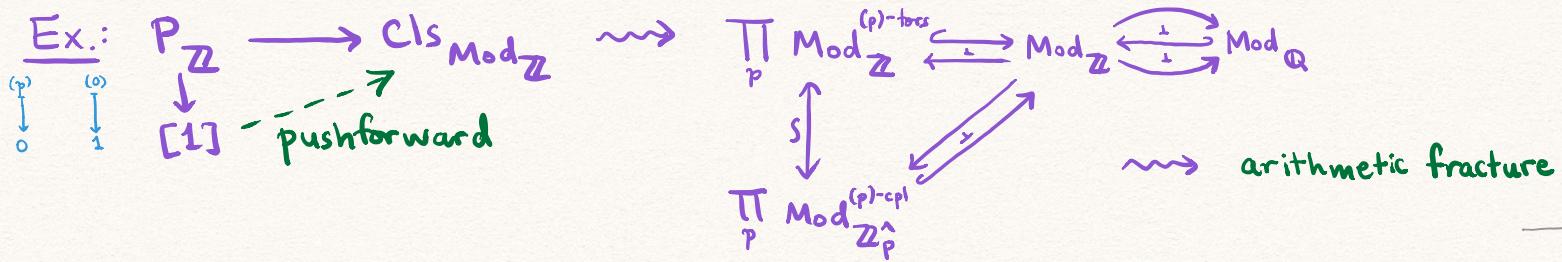
Moreover,  $(P \text{ down-finite}) \Leftrightarrow ((1) \text{ an equiv } ce) \Rightarrow ((2) \text{ an equiv } ce) \Rightarrow ((3) \text{ an equiv } ce) \Rightarrow ((4) \text{ an equiv } ce)$ .  
 $\forall p \in P, P_{\leq p} \text{ is finite}$

Q.: How to recover arithmetic fracture?

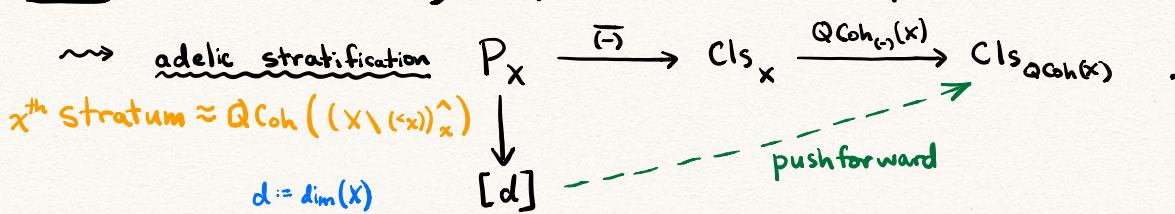
A.: Pushforward of stratifications.

THEOREM (fundamental operations):  $X$  stratified over  $P$ ;

- \*  $P \rightarrow Q$  a functor between posets  $\rightsquigarrow$  pushforward stratification of  $X$  over  $Q$ ;
- \*  $\forall p \in P$ ,  $X_p$  stratified over  $R_p$   $\rightsquigarrow$  refined stratification of  $X$  over  $P \downarrow R_p$ .
- \* also: restriction, pullback, quotient  
require condition of alignment ("point-set topology of closed subcats")



Ex.:  $X$  a scheme,  $P_X$  its specialization poset  $\rightsquigarrow P_X := |X|$  with  $x \leq y \Leftrightarrow x \in \bar{y}$



### §3: $\Theta$ -monoidal stratifications

$\Theta$  a nice operad (e.g.  $E_n$  for  $1 \leq n \leq \infty$ )

$R$  an  $\Theta$ -monoidal category

- $E_1$ : monoidal
- $E_2$ : braided monoidal
- $E_\infty$ : symmetric monoidal

Def<sup>n</sup>: A full subcat  $I \subseteq R$  is an ideal if it's contagious under  $\Theta$ -monoidal structure.

$$\rightsquigarrow I \xleftarrow[\text{(right-)laxly } \Theta\text{-monoidal}]{} R$$

$$\forall n \geq 2, \forall \mu \in \Theta(n), I \times R^{\otimes n-1} \xrightarrow{\otimes_{\mu}} I$$

Def<sup>n</sup>: A closed ideal is a closed subcat & ideal s.t.  $I \xleftarrow[\text{(strictly) } \Theta\text{-monoidal}]{} R$ .

Def<sup>n</sup>: An  $\Theta$ -monoidal stratification is a stratification that factors:  $P \xrightarrow{\quad} \text{Cls}_R$ .

$$\rightsquigarrow \text{strata } R_p := I_p / I_{\leq p} \text{ } \Theta\text{-monoidal}$$

$$R \xleftarrow[\text{laxly } \Theta\text{-monoidal}]{} R_p$$

$$, \quad P \xrightarrow[1.\text{lax}]{\mathcal{G}^\Theta(R)} \text{Alg}_\Theta(\text{Cat}) \xrightarrow{\text{fct}} \text{Alg}_\Theta(\text{Cat}) , \dots$$

$$\xrightarrow{x \mapsto \text{Id}_R} \text{Id}_R$$

full subposet on closed ideals

THEOREM ( $\Theta$ -monoidal reconstruction): Given an  $\Theta$ -monoidal stratification  $P \xrightarrow{\pi} \text{Id}_R$ ,

$$\exists \text{ canonical lift } R \xleftarrow[\lim_{sd(P)}^{\Theta}]{} \lim_{\text{lax}}^{\Theta} (\mathcal{G}^\Theta(R))$$

$$\text{Alg}_\Theta^{\text{lax}}(\text{Cat}) \xrightarrow{\text{fct}}$$

$$R \xrightleftharpoons[\lim_{\text{sd}(P)}]{\cong} \lim^{\text{r.lim}}_{\text{l.i.n.c.p.}} (\mathcal{G}(R))$$

Cat .

In particular,  $\mathcal{G}$  an equiv<sup>ce</sup> (e.g. if  $P$  down-finite)  $\Rightarrow \mathcal{G}^\otimes$  an equiv<sup>ce</sup>.

$$\text{Ex.: } P \xrightarrow{\mathbb{Z}_0} \text{Cls}_X \xrightarrow{\text{Qcoh}(X)} \text{Cls}_{\text{Qcoh}(X)}; \text{ so, strat}^n \text{ of } X \rightsquigarrow \text{sym. mon. strat}^n \text{ of } \text{Qcoh}(X)$$

$\downarrow$

Idf<sub>Qcoh(X)</sub>

Thm./Ex.:  $R$  rigidly-compactly generated sym. mon.  $\rightsquigarrow$  sym. mon. adelic strat<sup>n</sup> over  $\text{Spec}^{\text{Balmer}}(R^\omega)$ .

Thm. (closed ideals via idempotents):  $\text{Idf}_R \simeq \{ \underbrace{\text{central}}_{(\mathcal{I} \xleftrightarrow{i \circ j} R)} \underbrace{\text{augmented}}_{(i, j(1_R) \xrightarrow{\sim} 1_R)} \underbrace{\text{idempotents}}_{C \xrightarrow{\sim} 1_R \text{ s.t. } \forall n \geq 2, \forall \mu \in \mathcal{O}(n), \otimes_\mu(c, \dots, c) \xrightarrow{\sim} \otimes_\mu(1_R, c, \dots, c)}$ .

$$(I \xleftrightarrow{i \circ j} R) \mapsto (i, j(1_R) \xrightarrow{\sim} 1_R) \quad C \xrightarrow{\sim} 1_R \text{ s.t. } \forall n \geq 2, \forall \mu \in \mathcal{O}(n), \otimes_\mu(c, \dots, c) \xrightarrow{\sim} \otimes_\mu(1_R, c, \dots, c)$$

automatic for  $E_n, n \geq 2$

$$\left\{ \begin{array}{l} \forall n \geq 3, \forall \mu \in \mathcal{O}(n), \forall x_1, \dots, x_{n-2} \in R, \\ \otimes_\mu(c, c, x_1, \dots, x_{n-2}) \xrightarrow{\sim} \otimes_\mu(1_R, c, x_1, \dots, x_{n-2}) \end{array} \right.$$

$E_1: \text{equiv thg}, \forall x \in R, C \otimes x \xleftarrow{\sim} C \otimes x \xrightarrow{\sim} x \otimes C$

## §4: Equivariant cohomology

$G$  a finite group,  $M$  a  $G$ -manifold

\* for Poincaré duality, need  $\underline{R\mathcal{O}(G)}$ -graded cohomology:  $H_G^{i+v}(M) := H_G^{i+v}(M; \mathbb{Z}) \in \underline{\text{Mack}_G(\text{Ab})}$

$V \in K_0(\text{Rep}_{\mathbb{Z}}^{fd}(G))$

hard to compute, even for  $M = pt$ !

(for  $M = pt$ , known for  $G = C_p, C_{p^2}, C_{p^3}$ ; e.g. Hill-Hopkins-Ravenel)

Usual approach: homological algebra in  $\text{Mack}_G(\text{Ab})$ .

Key insight: Spaces  $\xrightarrow{\Sigma^\infty}$  Spectra  $\xrightarrow{(-) \otimes \mathbb{Z}}$   $\text{Mod}_{\mathbb{Z}}^{SG} \simeq \text{Mack}_G(\text{Mod}_{\mathbb{Z}})$

genuine  $G$ - $\mathbb{Z}$ -modules,  
a.k.a. "derived Mackey functors"

$H_G^{i+v}(-) \downarrow$

$\text{Mack}_G(\text{Ab}) \leftarrow \text{D}(\text{Mack}_G(\text{Ab})) \xrightarrow{\text{II}} \text{Mack}_G(\text{D}(\text{Ab}))$

the correct equivariant version of "chain complexes"

Thm. (sym. mon. stratification of  $\text{Mod}_{\mathbb{Z}}^{C_p^n}$ , after Kaledin and using Tate vanishing of Nikolaus-Scholze):

$$\text{Mod}_{\mathbb{Z}}^{gC_p^n} \xrightarrow{\sim} \lim \left( \text{Mod}_{\mathbb{Z}}^{hC_p^n} \xrightarrow{(-)^{t_{C_p}}} \text{Ar}(\text{Mod}_{\mathbb{Z}}^{hC_{p^{n-1}}}) \xrightarrow{(-)^{t_{C_p}}, \text{Mod}_{\mathbb{Z}}^{hC_{p^{n-2}}}} \cdots \xrightarrow{(-)^{t_{C_p}}, \text{Mod}_{\mathbb{Z}}^{hC_{p^{n-2}}}} \cdots \xrightarrow{(-)^{t_{C_p}}, \text{Mod}_{\mathbb{Z}}} \text{Ar}(\text{Mod}_{\mathbb{Z}}) \right).$$

as sym. mon. cats

Assume  $p$  odd, for slight simplification.

Cor.: The Picard group of  $\text{Mod}_{\mathbb{Z}}^{gC_p^n}$  is  $\mathbb{Z}^{\oplus(n+1)} \oplus \left( \bigoplus_{r=1}^n (\mathbb{Z}/p^{n-r+1})^\times / \{\pm 1\} \right) =: \mathbb{G}$ .

Cor.: The composite homomorphism  $R\mathcal{O}(C_p^n) \xrightarrow{V \mapsto S^V} \text{Pic}(S^{gC_p^n}) \xrightarrow{(-) \otimes \mathbb{Z}} \text{Pic}(\text{Mod}_{\mathbb{Z}}^{gC_p^n})$  is specified by its action on irreps:

SII

$$(C_{p^n} \xrightarrow{\text{triv}} O(1)) \longrightarrow ((1, 0, \dots, 0), (1, 1, \dots, 1))$$

$$(C_{p^n} \longrightarrow O(2)) \xrightarrow{\text{ker } J} ((2, 0, \dots, 0, \underbrace{-1, 0, \dots, 0}_{(k+2)^{\text{nd}} \text{ factor}}, (1, 1, \dots, 1)) .$$

Cor.: An explicit chain-level description of  $\text{Pic}(\text{Mod}_{\mathbb{Z}}^{g_{C_p^n}})$ -graded equivariant cohomology of pt,  $\forall n \geq 0$ .

and hence  
 $\text{RO}(C_p)$ -graded,  
by precomposition