

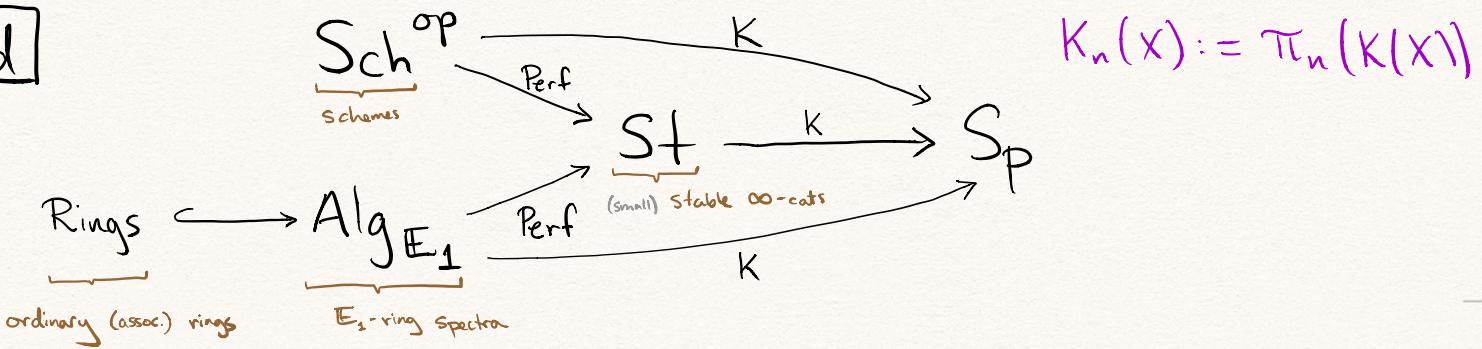
Secondary algebraic K-theory and traces

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conventions: implicit ∞ ; implicit $(-)^{\text{idem}}$; set theory in gray.

Outline

1d



Thm. [Blumberg-Gepner-Tabuada, v1]: The universal additive invariant: $St \xrightarrow{U} Mot$;
 and $\forall \mathcal{E} \in St$,

$$K(\mathcal{E}) \simeq \hom_{Mot}(\mathbb{1}, U(\mathcal{E})).$$

a sort of stabilization of St
 (ordinary stabilization = 0)

"K-theory is categorified stable homotopy":
 $\Sigma^\infty_+ \mathbb{S} \simeq \Sigma^\infty_+ \mathbb{S}$

cf. also Barwick (more general context)

$$S_* \longrightarrow Sp$$

$$X \mapsto \pi_*^{st}(X) \simeq \text{hom}_{Sp}(\mathbb{1}, \Sigma^\infty X)$$

~~~ trace maps

$$K(C) \xrightarrow{\text{tr}} \text{THH}(C) = \int_{S^1} C$$

$$K(C) \dashrightarrow TC(C)$$

cyclotomic spectrum

2d

$$\begin{array}{ccccc} Sch^{\text{op}} & \xrightarrow{\quad \text{Perf}^{(2)} \quad} & K^{(2)} & \xrightarrow{\quad \text{Perf}^{(2)} \quad} & St_2 \\ & \searrow & \downarrow & \swarrow & \downarrow \\ & & St_2 & \xrightarrow{\quad K^{(2)} \quad} & Sp \\ Alg_{E_2} & \xrightarrow{\quad \text{Perf}^{(2)} \quad} & \xrightarrow{\quad (\text{small}) \text{ stable } (\infty, 2)\text{-cats} \quad} & \xrightarrow{\quad K^{(2)} \quad} & \end{array}$$

"large"

$$St^L$$

$\Downarrow$

2-motives

Thm. [M-G-S]: The universal 2-additive invariant:  $St_2^\omega \xrightarrow{\mathcal{U}_2} Mot_2$ ;

and  $\forall X \in St_2^\omega$ ,

$$K^{(2)}(X) \simeq \text{hom}_{Mot_2}(\mathbb{1}, \mathcal{U}_2(X)).$$

~~~ trace maps

$$K^{(2)}(X) \xrightarrow{\text{tr}^{(2)}} \text{THH}^{(2)}(X) = \int_{T^2} X$$

$$K^{(2)}(X) \dashrightarrow TC^{(2)}(X)$$

2-cyclotomic spectrum

K⁽¹⁾

enhanced triangulated

Defⁿ: An ∞ -cat C is stable if:

↳ has finite limits & colimits,

↳ has a zero object $0_C := \overset{\text{initial}}{\underset{\sim}{\phi_C}} \rightsquigarrow \overset{\text{terminal}}{\underset{\sim}{pt_C}}$

↳ a sequence $A \xrightarrow{\quad} B \xrightarrow{\quad} C$ is fiber iff cofiber.

~~~ subcat  $\text{Fun}^{\text{ex}}(C, D) \subset \text{Fun}(C, D)$   
fully faithful = 0-full

~~~ sub-( $\infty, 2$ )-cat

$St \subset \underline{\text{Cat}}$

1-full
 $(\infty, 2)$ -cat of $(\infty, 2)$ -cats
(small)

Rmk.: $\text{Cat}(Sp) \xleftarrow[\text{fully faithful}]{} St$
spectrally enriched ∞ -cats
"stable envelope"



Ex.: $R \in Alg_E$, a ring spectrum $\rightsquigarrow Mod_R \in St^L$, $\text{Perf}(R) := Mod_R^\omega \in St$

Ex.: $X \in \text{Sch}$ a scheme $\rightsquigarrow \text{St}^L \ni Q\text{Coh}(X) \ni F: (\text{Spec}(R) \xrightarrow{\varphi} X) \mapsto \varphi^* F \in Q\text{Coh}(\text{Spec}(R)) := \text{Mod}_R$

$\text{St} \ni \text{Perf}(X) := Q\text{Coh}(X)^w$

Ex.: X a Δ space $\rightsquigarrow \text{Shv}_{\text{Sp}}^{\text{cb}}(X) \in \text{St}^L$ stratified

Analogy: $\text{St}_0 := \text{Sp}$, $\text{St}_1 := \text{St}$

$\hookrightarrow \text{St}$ has a zero object 0_{St} ... but now factorization $C \rightarrow 0_{\text{St}} \rightarrow D$ is merely a condition

$\hookrightarrow \text{St}$ is semiadditive: $C \times D \xleftarrow{\sim} C \amalg D =: C \oplus D$.

$\hookrightarrow \text{St}$ has a symm. mon. str.: $C \times D \longrightarrow C \otimes D$

$\text{St}_0 \rightsquigarrow \text{St}_1$
data \rightsquigarrow conditions [??]
not unlike e.g. (Ab, \otimes) !

$$C \longrightarrow D$$

$$\begin{array}{ccc} C \times D & \xrightarrow{\quad} & C \otimes D \\ \downarrow \text{AF} & & \downarrow \exists \leq 1 ? \\ C & & \end{array}$$

(and \exists iff F is biexact)

\hookrightarrow co/fiber sequences?

\rightsquigarrow 5+1 flavors of K-theory:

| flavor | "negative thinking" ☺ | direct sum K-theory | (connective) K-theory | nonconnective K-theory |
|---|---|--|--|--|
| π_0 | $\Sigma_{+}^{\infty} (i_0 C) \rightarrow \Sigma^{\infty} (i_0 C) \rightarrow K^{\oplus} (C)$
<small>\rightsquigarrow maximal sub-∞-gpds</small> | $\Sigma_{+}^{\infty} (i_0 C) \rightarrow \Sigma^{\infty} (i_0 C) \rightarrow K^{\oplus} (C) \rightarrow K(C)$
<small>\rightsquigarrow $[0_C] = 0$</small> | $K(C)$ | $K(C)$ |
| $\text{Thm. [BGT]}:$
corepresented
by \mathbb{I} in...
image of Sp^{fin} | $\text{obj}(C) \rightarrow \mathbb{Z}\{\text{obj}(C)\} \rightarrow \frac{\mathbb{Z}\{\text{obj}(C)\}}{[0_C] = 0} \rightarrow \frac{\mathbb{Z}\{\text{obj}(C)\}}{[A \oplus C] = [A] + [C]} \rightarrow \frac{\mathbb{Z}\{\text{obj}(C)\}}{[B] = [A] + [C] : \begin{cases} A \rightarrow B \\ 0_C \rightarrow C \end{cases}}$ | $\mathbb{Z}\{\text{obj}(C)\} / [0_C] = 0$ | $\mathbb{Z}\{\text{obj}(C)\} / [A \oplus C] = [A] + [C]$ | $\mathbb{Z}\{\text{obj}(C)\} / [B] = [A] + [C] : \begin{cases} A \rightarrow B \\ 0_C \rightarrow C \end{cases}$ |
| ... which is obtained from $\text{Fun}(\mathbb{I}, \text{St}^{\text{op}}, \text{Sp})$ by splitting... | $\Sigma_1 \text{St} \rightarrow \text{Fun}(\mathbb{I}, \text{St}^{\text{op}}, \text{Sp}) \xleftarrow{\sim} \frac{\text{Fun}}{[0_{\text{St}}] = 0} \xleftarrow{\sim} \frac{\text{Fun}}{[A \oplus C] = [A] + [C]}$ | $\text{Fun}(\mathbb{I}, \text{St}^{\text{op}}, \text{Sp}) \xleftarrow{\sim} \frac{\text{Fun}}{[0_{\text{St}}] = 0}$ | Mot^{add} | Mot^{loc} |

compact idempotent = complete

Thm [BGT]: $\forall A, B \in \text{St}$,

$\text{hom}_{\text{Mot}^{\text{add}}}(A, B) \simeq K(\text{Fun}^{\text{ex}}(A, B))$

$\{A \xrightarrow{i} A \oplus C \xrightarrow{j} C : A, C \text{ est}\}$
(Verdier) localization sequence := fiber & cofiber sequence

$$A \xleftarrow{i} B \xrightarrow{j} C$$

$$A \xleftarrow{R} B \xrightarrow{L} C$$

$$A \xleftarrow{i} B \xrightarrow{j} C$$

$$A \xleftarrow{R} B \xrightarrow{L} C$$

and e.g. $\text{Fun}^{\text{ex}}(\text{Sp}^{\text{fin}}, \mathcal{B}) \xrightarrow{\text{ev}_*} \mathcal{B}$

$$\text{hom}_{\text{Mot}^{\text{loc}}}(\mathcal{A}, \mathcal{B}) \simeq \mathbb{K}(\text{Fun}^{\text{ex}}(\mathcal{A}, \mathcal{B})).$$

s.t. $\forall b \in \mathcal{B}$, $iR(b) \xleftarrow{\varepsilon} b \xrightarrow{\eta} jL(b)$

$$b \simeq iR(b) \oplus jL(b)$$

$$\mathcal{B} \simeq \mathcal{A} \oplus \mathcal{C}$$

Ex.: $\mathbb{Z} \xrightarrow{\text{closed}} X \xleftarrow{\text{open}} \mathcal{U} \rightsquigarrow \text{Perf}_{\mathbb{Z}}(X) \hookrightarrow \text{Perf}(X) \rightarrow \text{Perf}(\mathcal{U})$

Ex.: $\mathcal{C} \xleftarrow{\perp} \text{OLoc}(\mathcal{C}) \xrightarrow{\perp} \mathcal{C}$

$\text{ev}_{\mathcal{C}/\mathcal{B}}$: $\text{Fun}([\mathcal{C}], \mathcal{C})$

O-localization seq. := co/fiber seq.

$A \xrightarrow{\quad} (A \xrightarrow{\varepsilon} A \rightarrow O_{\mathcal{C}}) \xrightarrow{\quad} (A \rightarrow B \rightarrow C) \xrightarrow{\quad} C$

$\eta \downarrow (O_{\mathcal{C}} \rightarrow C \xrightarrow{\varepsilon} C) \xrightarrow{\quad} C$

a O-loc. seq. in $\text{OLoc}(\mathcal{C})$

$\boxed{\text{tr}^{(1)}}$

Cor.: For any additive invariant $\text{St} \xrightarrow{F} \text{Sp}$, $\text{Nat}(K, F) \simeq F(\text{Sp}^{\text{fin}})$.

\approx Yoneda lemma

$O_{\text{St}} \xrightarrow{\quad} O_{\text{Sp}}$
 1-split 1-loc seq's \longmapsto O-loc. seq's := co/fiber seq's
 filt. colims \longmapsto filt. colims

Key Ex.: X a scheme,

$(\text{vbdl } E \downarrow X) \xleftrightarrow{\text{Bökstedt-Dennis trace}} \text{Sp}^{\text{ex}} \int_{S^1} \mathcal{C} \approx \left\{ \begin{array}{c} f_1 \\ \vdots \\ f_n \\ \text{co} \\ \vdots \\ f_{n+1} \end{array} \right\} / \sim \quad \mathcal{L}X := \text{map}(S^1, X)$

$\xrightarrow{(S^1 \xrightarrow{\gamma} X)} \left((S^1 \xrightarrow{\gamma} X) \mapsto \left(\begin{array}{c} \text{trace of} \\ \text{monodromy} \\ \text{of } \gamma^* E \downarrow S^1 \end{array} \right) \right)$

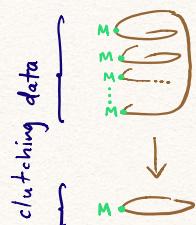
$K(X) \xrightarrow{\text{nil-invariant } / \mathbb{Q} \text{ [Goodwillie]}} \text{THH}(X) = \int_{S^1} \text{Perf}(X) \simeq \mathcal{O}(ZX)$

$\text{TC}^-(X) := \text{THH}(X)^{h\mathbb{T}} \simeq \mathcal{O}(ZX)^{h\mathbb{T}}$

$\text{TC}(X) := \text{THH}(X)^{h\text{Cyc}} \simeq \boxed{???$

Which functions on ZX ???

Idea: Given $r^* \gamma^* E \longrightarrow r^* E \longrightarrow E$, relation between $\text{tr}(r^* E)$ and $\text{tr}(r^* \gamma^* E)$.



e.g. $M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_d \end{pmatrix} \in \text{Mat}_{d \times d}(R)$

e.g. $r=2$:

$$\text{tr}(M)^{\otimes 2} - \text{tr}(M^{\otimes 2}) \in (R^{\otimes 2})^{C_2} \ni \sum_{i,j} m_i \otimes m_j - \sum_k m_k \otimes m_k$$

$$= \sum_{i < j} (m_i \otimes m_j + m_j \otimes m_i) = \sum_{i < j} \text{Norm}([m_i \otimes m_j])$$

$$(R^{\otimes 2})^{C_2} \ni [x \otimes y]$$

Norm

$$\sum_{\sigma \in C_2} \sigma(x \otimes y)$$

the Tate construction for $C_2 \cong R^{\otimes 2}$

$$\Rightarrow \text{tr}(M)^{\otimes 2} = \text{tr}(M^{\otimes 2}) \quad \text{in } (R^{\otimes 2})^{tC_2} := \text{cofib}(\text{Norm})$$

$$\rightsquigarrow \forall S_a^1 \leftarrow S_b^1, \text{ cyclotomic structure map } \text{THH}(e) = \left(\int_{S_a^1} e \right) \rightarrow \left(\int_{S_b^1} e \right)^{tC_2} = \text{THH}(e)^{tC_2}$$

[after Nikolaus-Scholze]

Thm. [A-M-G-R]: $\text{TC}(X)$
consists of those $f \in \mathcal{O}(\mathbb{L}X)$:

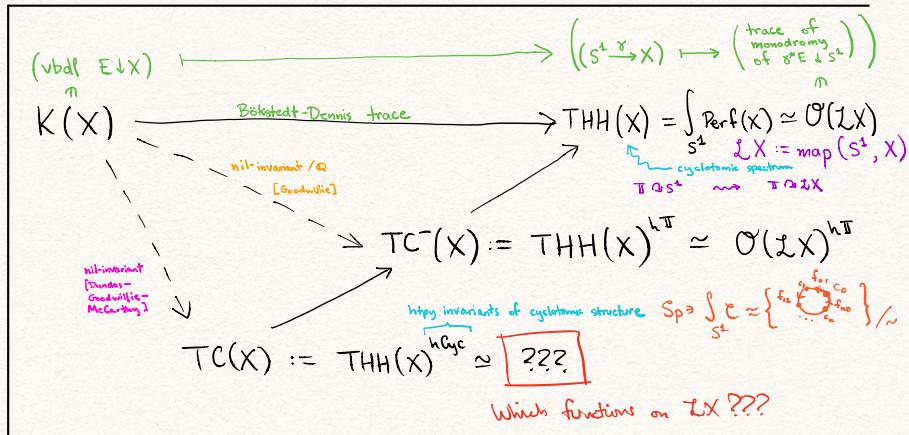
* \mathbb{T} -invariant;

$$\star \forall S^1 \xrightarrow{r} S^1 \xrightarrow{\tau} X$$

$\rightsquigarrow \tau, r^* \tau \in \mathbb{L}X$, we have

$$f(r^* \tau) \underset{\text{modulo universal indeterminacy (i.e. norms)}}{\equiv} f(\tau)^r.$$

I.e., precisely the structure present on
 $\text{tr.man}(E) \in \mathcal{O}(\mathbb{L}X)$ for vector bundles $E \downarrow X$!



$K^{(2)}$

prehistory: CFT, 2-vector spaces, 2-vector bundles [Segal; Kapranov, Voevodsky; Baas, Dundas, Rognes]

around $K^{(2)}$: motives, motivic measures, $K(\text{Var})$, Brauer groups [Kontsevich; Cisinski, Tabuada; Bondal, Larsen, Lunts; Toën, Vezzosi]
higher-dim. YBE chromatic ht 2
point counts/ \mathbb{F}_q ...

around $\text{tr}^{(2)}$: higher character theory, (categorified) trace & index theorems [Ben-Zvi, Nadler; Ganter, Kapranov; Hopkins, Safarov; Atiyah-Bott-Lefschetz, Grothendieck-Riemann-Roch, ... Scherotzke, Sibilla; Carlsson, Douglas, Dundas ...]

* chromatic homotopy theory:

| chromatic height | prototypical coh. thy. | ... whose cocycles are families of... | ... which are objects of... |
|------------------|---|---------------------------------------|--------------------------------|
| 0 | $H\mathbb{Q}^*$
rational coh. | rational numbers | the 0-cat \mathbb{Q} |
| 1 | KU^*
cx. K-theory | \mathbb{C} -vector spaces | the 1-cat Vect $_{\mathbb{C}}$ |
| 2 | Ell^*
elliptic coh.
(& tmf = topological modular forms) | ??? | ??? |

cf. Stolz, Teichner; Berwick-Evans, Tripathy

Idea: $K^{(2)}$ a height-2 coh. thy. in algebraic geometry.

height-0: e.g. étale coh.

height-1: e.g. K-theory

Defⁿ: A quasicoherent sheaf of categories over a scheme X is

$$QCoh^{(2)}(X) \ni \mathcal{C} : (\text{Spec}(R) \xrightarrow{\varphi} X) \longmapsto \varphi^*\mathcal{C} \in QCoh^{(2)}(\text{Spec}(R)) := St_R$$

$$\text{Perf}^{(2)}(X) := QCoh^{(2)}(X)^\omega$$

$R = \text{linear stable } \infty\text{-cats}$
(small)

Rmk.: Often, $QCoh^{(2)}(X) \xrightarrow{\sim} \text{Mod}_{QCoh(X)}(St^L)$. "X is 1-affine"
[Gaitsgory]

Provisional Defⁿ: An $(\infty, 2)$ -cat X is stable if it satisfies the conditions:

↪ it's enriched over $(St, \otimes) \subset (\text{Cat}, \times)$;

↪ it has finite products. \Rightarrow zero object $0_X := \emptyset_X \xrightarrow{\sim} pt_X$
 $:= St\text{-enr. functors}$

semiadditive: $\oplus := x \xleftarrow{\sim} \perp \perp$

same as Ab-enriched
cats: for $A \in X \in \text{Cat}(St)$,
 A is terminal
iff A is initial
iff $\text{hom}_X(A, A) \simeq 0_{St}$
(pf.: $\text{Mod}_{0_{St}}(St) = 0$)

$$\rightsquigarrow \underbrace{\text{Fun}^{2-\text{ex}}(X, Y)}_{\text{fully faithful}} \subset \text{Fun}(X, Y)$$

$$\rightsquigarrow \text{sub-}(\infty, 3)\text{-cat } St_2 \subset \text{Cat}_2 \text{, also } \underbrace{\text{Cat}_2(Sp)}_{\text{fully faithful}} \xleftrightarrow{\perp} \text{Cat}(St) \xleftrightarrow{\perp} St_2.$$

$(\infty, 2)\text{-cats enr. in } (Sp, \otimes) =: (St_0, \otimes)$
 $\approx \text{"dg 2-cats"}$

Guess: Final defⁿ should involve localization sequences.

Q.: Defⁿ not making reference to St ? ($St \subset \text{Cat}$ defined without Sp .)

Rmk: Thm. below holds for any defⁿ \widetilde{St}_2 , so long as $St_2 \xleftarrow[\text{ff.}]{\perp} \widetilde{St}_2$.

Ex.: $X \in \text{Sch}$ a scheme $\rightsquigarrow QCoh^{(2)}(X) \in St_2^L$, $\text{Perf}^{(2)}(X) \in St_2$

Ex.: $R \in \text{Alg}_{E_2}$ $\rightsquigarrow St_R \in St_2^L$, $\text{Perf}^{(2)}(R) := St_R^\omega \in St_2$

Ex.: X a ^{stratified} space $\rightsquigarrow Shv_{St}^{\text{cb}}(X) \in St_2^L$

Ex.: Symp: $\text{hom}(M, N) := \text{Fuk}(\overline{M} \times N)$
modulo analysis!

Defⁿ: A (1-split) 1-localization sequence in $X \in St_2$ is

$$A \xrightleftharpoons[\substack{i \\ R}]{} B \xrightleftharpoons[\substack{L \\ j}]{} C \quad \text{s.t.}$$

$\hookrightarrow L_i = 0$ in $\hom_X(A, C)$ $\Leftrightarrow R_j = 0$ in $\hom_X(C, A)$

$$\begin{array}{ccc} iR & \xrightarrow{\varepsilon} & \text{id}_B \\ \downarrow & \square & \downarrow \\ 0 & \longrightarrow & jL \end{array} \quad \text{a } \underbrace{0\text{-loc. sequence}}_{\text{i.e. co/fiber seq.}} \text{ in } \hom_X(B, B) \in St.$$

$$\rightsquigarrow z_1 X \longrightarrow \text{Fun}(z_1 X^{\text{op}}, \mathbf{Sp}) \xrightleftharpoons[\substack{\text{Cat} \\ \text{localization: } 0_X \mapsto 0_{\mathbf{Sp}} \\ 1\text{-loc's} \mapsto 0\text{-loc's}}]{} \text{Mot}(X) \in St^L \quad \text{motives over } X$$

$$\begin{aligned} K^{(2,1)}(X) &:= \text{Mot}(X)^{\omega} \\ \rightsquigarrow K^{(2,0)} &:= K^{(1,0)} \circ K^{(2,1)} \end{aligned}$$

Defⁿ: The secondary K-theory of $X \in St_2$ is $K^{(2)}(X) := K(\text{Mot}(X)^{\omega})$.

Rmk.: Really, \exists many flavors of $K^{(2)}$: choose flavors of Mot & of K.

[This is the simplest non-silly one.]

Rmk.: For $R \in \text{Alg}_{\mathbb{E}_2}$, have $K(K(R)) \longrightarrow K^{(2)}(R)$. \star redshift ???

Defⁿ: A (2-split) 2-localization sequence in St_2 is

$$X \xrightleftharpoons[\substack{i \\ R}]{} Y \xrightleftharpoons[\substack{L \\ j}]{} Z \quad \text{s.t.}$$

$\hookrightarrow L_i = 0$ in $\text{Fun}^{2\text{-ex}}(X, Z)$ $\Leftrightarrow R_j = 0$ in $\text{Fun}^{2\text{-ex}}(Z, X)$

$$\begin{array}{ccccc} iR & \xrightarrow{\varepsilon} & \text{id}_Y & \xleftarrow{\eta} & jL \\ \dashleftarrow & \dashleftarrow & \dashleftarrow & \dashleftarrow & \dashleftarrow \\ \varepsilon \varepsilon^R & \longrightarrow & \text{id}_{\text{id}_Y} & \longrightarrow & \eta^R \eta \end{array} \quad \text{a } \underbrace{1\text{-loc. sequence}}_{\text{in } \text{Fun}^{2\text{-ex}}(Y, Y) \in St_2} \text{ in } \text{Fun}^{2\text{-ex}}(Y, Y)$$

st. $\varepsilon \varepsilon^R \longrightarrow \text{id}_{\text{id}_Y} \quad \text{a } 0\text{-loc. seq. in } \hom_{\text{Fun}^{2\text{-ex}}(Y, Y)}(\text{id}_Y, \text{id}_Y) \in St$

$$\text{Ex.: } X \xrightleftharpoons{\pm} 1\text{Loc}(X) \xrightleftharpoons{\pm} X$$

Almost-ex.: $Z \xrightarrow{\text{closed}} X \xleftarrow{\text{open}} U \rightsquigarrow \text{Perf}_Z^{(2)}(X) \hookrightarrow \text{Perf}^{(2)}(X) \rightarrow \text{Perf}^{(2)}(U)$

only 0-split \rightsquigarrow need a better (nonconn.) version of $K^{(2)}$!

$$\rightsquigarrow \iota_2 \text{St}_2 \longrightarrow \text{Fun}(\iota_2 \text{St}_2^\omega, \text{St}) \xrightleftharpoons[\perp]{\quad} \text{Mot}_{2,1} \in \text{St}_2^L$$

$\begin{array}{c} \text{Cat}_2^L \\ \uparrow \end{array}$

localization: $\mathcal{O}_{\text{St}_2} \longmapsto \mathcal{O}_{\text{St}}$

2-loc's \longmapsto 1-loc's
all values idempotent-complete

Thm. A (after BGT): $\forall A, B \in \mathcal{X} \in \text{St}_2, \text{hom}_{\text{Mot}(X)}(A, B) \simeq K(\text{hom}_X(A, B))$.

Thm. B: $\forall X, Y \in \text{St}_2$, $\text{hom}_{\text{Mot}_{2,1}}(X, Y) \simeq \text{Mot}(\text{Fun}^{2-\text{ex}}(X, Y))^\omega$.

Def": $\text{Mot}_2 := \text{Mot}(\text{Mot}_{2,1}^\omega)$ $\rightsquigarrow \iota_1 \text{St}_2 \longrightarrow \iota_1 \text{Mot}_{2,1} \longrightarrow \text{Mot}_2$

$\begin{array}{c} \text{compact} \\ \downarrow \\ \text{2-motives} \end{array}$

Cor. (A+B): $\forall X, Y \in \text{St}_2^\omega, \text{hom}_{\text{Mot}_2}(X, Y) := \text{hom}_{\text{Mot}(\text{Mot}_{2,1}^\omega)}(X, Y)$

$\simeq K(\text{hom}_{\text{Mot}_{2,1}}(X, Y)) \stackrel{\text{Thm. B}}{\simeq} K(\text{Mot}(\text{Fun}^{2-\text{ex}}(X, Y))) =: K^{(2)}(\text{Fun}^{2-\text{ex}}(X, Y))$. □

tr⁽²⁾

$K^{(2)}(X) \xrightarrow{\text{tr}^{(2)} \circ 0} \text{THH}^{(2)}(X) = \int_{T^2} \text{Perf}^{(2)}(X) \simeq \mathcal{O}(L^2 X)$

$\text{nil-invariant } ?? \rightsquigarrow \text{TC}^{(2)}(X) \simeq \begin{cases} f \in \mathcal{O}(L^2 X) \text{ w/ compatibility-mod-norms} \\ \forall \text{framed coverings } T^2 \xleftarrow{\quad} T^2 \end{cases}$

$\nexists \exists \text{ nontrivial space of framing-preservation data, even just for } T^2 \xleftarrow{\text{id}} T^2 !$

$\text{Aut}^{\text{fr}}(T^2) \simeq (\mathbb{P}^2 \times \mathbb{Z}) \times \text{SL}_2(\mathbb{Z}), \text{End}^{\text{fr}}(T^2) \simeq \dots$