

DERIVED MACKEY FUNCTORS AND

C_{p^n} - EQUIVARIANT COHOMOLOGY

VIA STRATIFICATIONS

↳ a new formalism for computations, plus examples –
many more computations to be done!

joint with David Ayala & Nick Rozenblum

cf. "Derived Mackey functors...", arxiv: 2105.02456

(and "Stratified noncommutative geometry", arxiv: 1910.14602)

these slides: etale.site/writing/eCHT-mackey.pdf

§1 : Eqvrt coh. via strat" of genuine G-spectra

§2 : " " derived Mackey functors

§ 3: Computations

§ 1: G -spectra

G a finite group

nonequivariant:

$$\begin{array}{ccc} \text{Spaces} & \xrightarrow{H^i} & \text{Ab}^\text{op} \\ \Sigma_+^\infty \downarrow & & \nearrow \\ \text{Spectra} & \dashrightarrow & \end{array}$$

$$H^i(X) := H^i(X; \mathbb{Z}) \cong \left[\Sigma_+^\infty X, S^i \otimes H\mathbb{Z} \right]_{Sp}$$

Eilenberg-MacLane

equivariant: \wedge ^{genuine!} G -Spaces

$$\begin{array}{ccc} & H_G^{i+v} & \\ \dashrightarrow & \xrightarrow{\quad} & \text{Mack}_G(\text{Ab})^\text{op} \\ & \nearrow & \end{array}$$

Mackey functors for G

* keep track of "strict" fixedpoints for all $H \subseteq G$

$$\sum_{+,G}^{\infty} \downarrow$$

genuine!

\wedge **G-Spectra**

$\vdash := \{ \text{subgps} \} \rightarrow \text{Ab}$

w/ covariant & contravariant factoriality,
transfer restriction
conjugation iso's, double coset formula...

$$:= \text{Fun}^{\oplus}(\underbrace{\text{Burn}_G}_{\text{Ab}}, \text{Ab})$$

the Burnside (2,1)-category of G :

$$\begin{cases} \text{ob} = \text{finite } G\text{-sets} \\ \text{mor} = \text{spans} \\ 2\text{-mor} = \text{iso's betw. spans} \end{cases}$$

$$\begin{aligned} H \subseteq K \subseteq G \rightsquigarrow G/H \rightarrow G/K \text{ in } \text{Fin}^G \\ \rightsquigarrow \left(\begin{array}{c} G/H \\ G/K \\ \swarrow \quad \searrow \\ G/H \end{array} \right) \longrightarrow \left(H_G^{i+v}(X)(K) \xrightarrow{\text{res}_H^K} H_G^{i+v}(X)(H) \right) \\ \left(\begin{array}{c} G/H \\ \swarrow \quad \searrow \\ G/K \end{array} \right) \longrightarrow \left(H_G^{i+v}(X)(H) \xrightarrow{\text{trf}_H^K} H_G^{i+v}(X)(K) \right) \\ \rightsquigarrow \text{Burn}_G \qquad \qquad \qquad \text{Ab} \end{aligned}$$

$$H_G^{i+v}(X) := H_G^{i+v}(X; \underline{\mathbb{Z}}) \cong \left(G/H \longmapsto \left[\sum_{+,G}^{\infty} X \otimes \sum_{+,G}^{\infty} (G/H), \mathcal{S}^i \otimes \mathcal{S}' \otimes H \underline{\mathbb{Z}} \right]_{\text{Sp}^G} \right)$$

Eilenberg-MacLane

$V \in \text{RO}(G)$ a virtual rep $\rightsquigarrow \mathcal{S}^V \in \text{Pic}(\text{Sp}^G)$

$$\hookrightarrow V \text{ a } G\text{-rep} \rightsquigarrow \mathcal{S}^V = \sum_G^{\infty} (V^+)$$

$$\hookrightarrow V = [V_+, V_-] \rightsquigarrow \mathcal{S}^V = \mathcal{S}^{V_+} \otimes (\mathcal{S}^{V_-})^{\otimes -1}$$

$$\text{e.g. } H_G^{i+v}(\text{pt}) \cong \left(G/H \longmapsto \pi_0 \left((\mathcal{S}^i \otimes \mathcal{S}' \otimes H \underline{\mathbb{Z}})^H \right) \right)$$

"the $(i+v)^{\text{th}}$ G -equivrt coh. of a point"

"categorical H -fixedpts", $(-)^H = \text{hom}(\sum_{+,G}^{\infty} (G/H), -)$

Thm. (Guillou-May, Barwick): $\text{Sp}^G \simeq \text{Mack}_G(\text{Sp}) := \text{Fun}^{\oplus}(\text{Burn}_G, \text{Sp})$.

$$E \rightsquigarrow (G/H \mapsto E^H)$$

\cong
hom($\Sigma_{+c}(G/H), E$)

Problem: \otimes of G -spectra \ncong pointwise \otimes of spectral Mackey fcts!

i.e. $(E \otimes F)^H \neq E^H \otimes F^H$; $(-)^H$ isn't monoidal.

There are also the geometric fixedpoints, $\Phi^H(E) := (E \otimes \widetilde{EF}_{\neq H})^H$.

These fcts are symmetric monoidal, and:

Thm. (Greenlees, "geom. fp's Whitehead"): If $\Phi^H(E \rightarrow F)$ an equiv^{ce} $\forall H \subseteq G$, then $E \rightarrow F$ an equiv^{ce}. I.e. $\{\Phi^H\}_{H \subseteq G}$ are jointly conservative.

Warning: The underlying "spectrum w/ homotopy G -action" fctr

$$Sp^G \xrightarrow[\perp]{U} Sp^{hG} := \text{Fun}(BG, Sp)$$

is not conservative.

β

inclusion of "Borel-complete" G-spectra: $E^H \xrightarrow{\sim} E^{hH}$

unstably, e.g. $(EG)_{SI}^G \xrightarrow{\sim} (EG)_{SI, pt}^{hG}$

Hint of solution: use geom. f.p.'s instead of categorical f.p.'s as an alternative "basis" for Sp^G (compat. w/ \otimes).

Ex. (Greenlees-May): A genuine C_p -spectrum $E \in Sp^{C_p}$ is completely recorded by:

$$\textcircled{1} \quad UE \simeq \Phi^e E \in Sp^{hC_p},$$

$$\textcircled{2} \quad \Phi^{C_p} E \in Sp,$$

$$\textcircled{3} \quad \Phi^{C_p} E$$

↓ "gluing data" $\Phi^{C_p}(\eta_{u+\beta}(E))$ in Sp

$$\underbrace{\Phi^{C_p}}_{\beta}(UE) = (UE)^{+C_p}$$

the C_p -Tate constrn: for $A \in Sp^{hC_p}$,

$$A_{ncp} \xrightarrow{\text{Norm}} A^{hC_p} \xrightarrow{\text{cofib.}} A^{+C_p}$$

$$\text{I.e., } \mathbf{Sp}^{C_P} \simeq \lim^{\text{r.lax}} \left(\mathbf{Sp}^{hC_P} \xrightarrow{(-)^{tC_P}} \mathbf{Sp} \right)$$

right-lax limit:

$$\lim^{\text{r.lax}} (c \xrightarrow{F} D) := \left\{ \left(c_1 \xrightarrow{\begin{smallmatrix} d \\ \downarrow F_c \end{smallmatrix}} \right) \right\} \simeq \lim (c \xrightarrow{\begin{smallmatrix} \text{Ar}(D) \\ \downarrow \perp \end{smallmatrix}} D)$$

$$\lim (c \xrightarrow{F} D) := \left\{ \left(c_1 \xrightarrow{\begin{smallmatrix} d \\ \downarrow F_c \end{smallmatrix}} \right) \right\} \simeq c$$

$$\lim^{\text{l.lax}} (c \xrightarrow{E} D) := \left\{ \left(c_1 \xrightarrow{\begin{smallmatrix} d \\ \uparrow F_c \end{smallmatrix}} \right) \right\}$$

$$\text{Cor.: } \hom_{\mathbf{Sp}^{C_P}}(F, E) \simeq \lim \left(\begin{array}{c} \hom_{\mathbf{Sp}}(\Phi^{C_P} F, \Phi^{C_P} E) \\ \downarrow \\ \hom_{\mathbf{Sp}^{hC_P}}(UF, UE) \longrightarrow \hom_{\mathbf{Sp}}(\Phi^{C_P} F, (UE)^{tC_P}) \end{array} \right).$$

$$\text{e.g. } F = \mathbb{S} = \varprojlim_{t, C_P} (c_p / c_p) \rightsquigarrow E^{C_P} \simeq \lim \left(\begin{array}{c} \Phi^{C_P} E \\ \downarrow \\ (UE)^{hC_P} \longrightarrow (UE)^{tC_P} \end{array} \right).$$

$$\left((-)_{hC_P} \xrightarrow{\text{fib}} (-)^{C_P} \rightarrow \Phi^{C_P} \right) \left(\begin{array}{c} E \\ \downarrow \\ \beta_{UE} \end{array} \right) = \left(\begin{array}{c} E_{hC_P} \xrightarrow{\text{fib}} E^{C_P} \xrightarrow{\Phi^{C_P} E} \\ \parallel \\ E_{hC_P} \xrightarrow{\text{fib}} E^{hC_P} \xrightarrow{\downarrow} E^{tC_P} \end{array} \right)$$

"isotropy separation"

Thm. (AMR, after Glasman):

↪ P_G the poset of subqps of G , ordered by subconjugacy

↪ for $H \in P_G$, $W(H) := \frac{N(H)}{H}$ its Weyl group
 $\cong G/H$ if G abelian (or $H \trianglelefteq G$)
 $\cong \text{hom}_G(G/H, G/H)$

Then, $Sp^G \xrightarrow{\sim} \lim^{r.\text{lax}} \left(P_G \xrightarrow{\text{l.lax}} \text{Cat} \right)$

\Downarrow $\Downarrow H \mapsto Sp^{hW(H)}$

$E \mapsto \left\{ \Phi^H(E) \in Sp^{hW(H)} \right\}_{H \in P_G}$

+ gluing data

e.g. $\Phi^{Sp^G} \downarrow (\Phi^E)^{t_{CP}}$

Ex.: $G = C_{p^2}$, $P_{C_{p^2}} = \left\{ e \xrightarrow{\begin{smallmatrix} \text{c}_p \\ \text{c}_p \end{smallmatrix}} C_{p^2} \right\}$,

$Sp^{C_{p^2}} \xrightarrow{\sim} \lim^{r.\text{lax}} \left(\begin{array}{ccc} & Sp^{hC_p} & \\ & \uparrow & \\ Sp^{hC_{p^2}} & \xrightarrow{\quad (-)^{t_{CP}} \quad} & Sp \\ & \uparrow & \\ & (-)^{t_{CP}} & \end{array} \right)$

$\left\{ \begin{array}{c} E_1 \\ \downarrow \tau_{01} \\ E_0^{t_{CP}} \end{array}, \quad \left\{ \begin{array}{c} E_2 \\ \downarrow \tau_{02} \\ E_0^{t_{CP}} \end{array} \xrightarrow{\tau_{12}} \left(\begin{array}{c} E_1^{t_{CP}} \\ \downarrow \tau_{01}^{t_{CP}} \end{array} \right) \right\} \right\}$

the proper Tate constrⁿ: quotient by norms from all proper subgroups

Thm.: can also be defined as $(-)^{t_G}: Sp^{hG} \xrightarrow{\beta} Sp^G \xrightarrow{\Phi} Sp$.

"higher-categorical gluing data"

e.g. Cor.: $E^{C_{P^2}} \simeq \lim$

$$E_i := \bigoplus^{C_{P^i}}(E)$$

$$\begin{array}{ccccc} E_2 & \xrightarrow{\quad} & E_1^{t_{CP}} & & \\ \downarrow & & \downarrow & & \\ E_0^{t_{CP^2}} & \xrightarrow{\quad} & E_1^{h_{CP}} & \xrightarrow{\quad} & (E_0^{t_{CP}})^{t_{CP}} \\ & \nearrow & \downarrow & \nearrow & \\ E_0^{h_{CP^2}} & \xrightarrow{\quad} & (E_0^{t_{CP}})^{h_{CP}} & \xrightarrow{\quad} & \end{array}$$

$$\begin{array}{ccccc} 2 & \xrightarrow{\quad} & 12 & & \\ \downarrow & & \downarrow & & \\ 02 & \xrightarrow{\quad} & 1 & \xrightarrow{\quad} & 012 \\ \uparrow & & \downarrow & & \\ 0 & \xrightarrow{\quad} & 01 & \xrightarrow{\quad} & \end{array}$$

Rmk.: This comes from a (sym. mon.) Stratification of $\chi = Sp^G$ over the poset $P = P_G$.

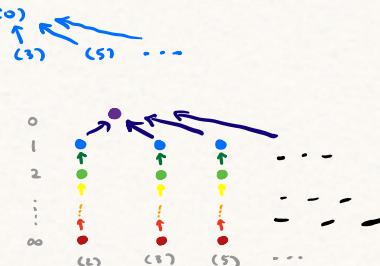
Formally analogous to e.g.:

↪ $\chi = QCoh(X)$ for X a P -strat^d scheme;

e.g. $X = \text{Spec}(Z) \rightsquigarrow$ adelic strat over $\begin{matrix} (0) \\ (2) \\ (3) \\ (5) \\ \dots \end{matrix}$
 ↦ record $M \in \text{Mod}_Z$ in terms of $\text{rat}^m \otimes_Z M$, $p\text{-cpl}^n M_p$, and gluing data

↪ $\chi = Sp \rightsquigarrow$ chromatic strat over

↪ $\chi = Shv(T)$ for T a P^o -strat^d top. spc.



$$\text{Ex.: } Sp^{S_3} \simeq \lim_{\substack{\text{r.lax} \\ \text{ZERO!}}} \left(\begin{array}{ccc} Sp^{n_{S_3}} & \xrightarrow{(-)} & Sp \\ \downarrow (-)^{t_{C_3}} & \searrow & \downarrow \\ Sp^{hC_2} & \xrightarrow{(-)^{t_{C_2}}} & Sp \end{array} \right) \simeq \left\{ \left(\begin{array}{c} E_{00} \xleftarrow{(-)} E_{00} \xleftarrow{(-)^{t_{C_2}}} E_{01} \\ \downarrow E_{00}^{t_{C_3}} \\ E_{10} \xleftarrow{(-)} E_{10} \xleftarrow{(-)^{t_{C_2}}} E_{11} \end{array} \right) \right\}.$$

Ex.: $Sp^{< S^2}$, cyclotomic spectra, ...

§2: Derived Mackey functors

Spectra are good for generalized coh thys! But for ordinary (singular), can just use chain cxes, i.e. the derived (∞) -cat $D(\mathbb{Z})$. —

$$\begin{array}{ccc} \text{Spaces} & \xrightarrow{H^i} & Ab^{\oplus} \\ \Xi^{\infty}_+ \downarrow & \nearrow & \\ \text{Spectra} & & \end{array}$$

$$\begin{array}{ccc} \text{G-Spaces} & \xrightarrow{H^{i+n}} & \text{Mack}_G(Ab)^{\oplus} \\ \Xi^{\infty}_{+,G} \downarrow & \nearrow & \\ \text{Mack}_G(Sp) \simeq G\text{-Spectra} & & \text{e.g.} \end{array}$$

$\text{Mod}_{H\mathbb{Z}}^{S^1}(Sp)$

i.e. evaluate at
G/H e Burns

$$\begin{array}{c}
 H\mathbb{Z} \otimes (-) \downarrow \text{C}_*^{\text{sing}} \\
 D(\mathbb{Z}) \\
 \text{Mack}_G(H\mathbb{Z} \otimes (-)) \downarrow \\
 \text{Mack}_G(D(\mathbb{Z})) =: \text{DMack}_G \\
 \downarrow \\
 H_G^{i+\nu}(\text{pt}) \cong \left(G/H \mapsto H_0((\mathbb{Z}^{i+\nu} \otimes_{\mathbb{Z}} \mathbb{Z})^H) \right)
 \end{array}$$

derived Mackey ftrs for G

Kaledin : $\text{DMack}_G = \text{Mack}_G(D(\mathbb{Z}))$ ☺

$D(\text{Mack}_G(\text{Ab}))$ ☹ ← resolutions of Mackey ftrs; "off by a spectral sequence"

Problem (again) : $\text{DMack}_G \xrightarrow{(-)^H} D(\mathbb{Z})^{hW(H)}$ not monoidal .

Solution (again) : alternative sym. mon. "basis" via Φ^H .

$$\text{Thm. : } \text{DMack}_G \xrightarrow{\sim} \lim^{\text{r.lax}} \left(P_G \xrightarrow{\text{l.lax}} \text{Cat} \right) .$$

\Downarrow \Downarrow
 $H \longmapsto D(\mathbb{Z})^{hW(H)}$

Again, gluing ftrs are proper Tate constrns.

Fact : $\text{Sp} \xrightarrow{H\mathbb{Z} \otimes (-)} D(\mathbb{Z})$ does not commute w/ Tate!

Indeed, Tate in $D(\mathbb{Z})$ much simpler than Tate in Sp .

↪ not a bug, but a feature!

E.g. in $D(\mathbb{Z})$, $((-)^{t_{C_p}})^{t_{C_p}} = 0$ and $(-)^{t_{C_p^2}} = 0$.

More generally, in "strat" of $DMack_{C_p^n}$ over $P_{C_p^n} = \{e \rightarrow c_p \rightarrow \dots \rightarrow c_p^m\}$,

- ↪ all non-adjacent gluing fctrs are zero;
- ↪ all composites of gluing fctrs are zero.

↪ $DMack_{C_p^n} \simeq \left\{ \left(\begin{array}{c} E_0 \xrightarrow{\quad} E_1 \xrightarrow{\quad} \dots \xrightarrow{\quad} E_n \\ \underbrace{\quad}_{D(\mathbb{Z})^{hC_p^n}} \quad \underbrace{\quad}_{D(\mathbb{Z})^{hC_p^{n-1}}} \dots \underbrace{\quad}_{D(\mathbb{Z})^{hC_p^1}} \dots \underbrace{\quad}_{D(\mathbb{Z})} \end{array} \right) \right\}$

e.g. Cor.: $E^{C_p^2} \simeq \lim \left(\begin{array}{ccc} E_2 & \longrightarrow & E_1^{t_{C_p}} \\ \downarrow & \nearrow & \downarrow \\ E_0 & \xrightarrow{t_{C_p^2}} & (E_0^{t_{C_p}})^{hC_p} \end{array} \right)$

$E_i := \Phi^{C_p^i}(E)$

and more generally, $E^{C_{p^n}} \simeq \lim \left(E_0^{hC_{p^n}} \xrightarrow{E_1^{hC_{p^{n-1}}} \dots \xrightarrow{E_n^{hC_{p^1}}} (E_0^{tC_p})^{hC_{p^{n-1}}} \right)$.

This makes computations algorithmic! Simple enough that even I can do them.
 still nontrivial
 but now
 in particular, no spectral sequences

§3: Computations

p an odd prime

$$\text{Cor. 1: } \text{Pic}(\text{DMack}_{C_{p^n}}) \cong \mathbb{Z}^{\oplus(n+1)} \oplus \left(\bigoplus_{s=2}^n (\mathbb{Z}/p^{n-s+1})^\times / \{\pm 1\} \right).$$

Thm. (Krause): $\text{Pic}(Sp^G) \xrightarrow{!} \text{Pic}(\text{DMack}_G)$.

Pf. sketch: ① $\text{DMack}_{C_{p^n}} \xrightarrow{\Phi^{C_{p^n}}} D(\mathbb{Z})^{hC_{p^{n-1}}}$ is sym mon., so

carries Picard elts to Picard elts.

② Since p odd, $\text{Pic}(D(\mathbb{Z})^{hC_{p^{n-1}}}) \cong \{\mathbb{Z}^\times\}_{n \in \mathbb{Z}} \cong (\mathbb{Z}, +)$.

③ Determine which gluing data among these give
 Tate coh. classes
 Picard elts (using sym. monoidality of strat").

Q.: Other groups?

e.g. dihedral, symmetric, alternating...

Krause: various partial results

Angeltveit (more recently): $C_k \rtimes k$

Cor. 2: The homomorphism

freely gen^d by
irreps, and:

$$\begin{array}{ccc}
 RO(C_{p^n}) & \xrightarrow{\quad V \mapsto \mathbb{Z}^V \quad} & \text{Pic}(Sp^{C_{p^n}}) \\
 \uparrow \text{Rep}(C_{p^n}) & \nearrow \text{dashed} & \downarrow \text{H}\mathbb{Z} \otimes (-) \\
 & & \text{Pic}(DMack_{C_{p^n}}).
 \end{array}$$

Cor. 1: $\text{Pic}(DMack_{C_{p^n}}) \cong \bigoplus_{i=1}^{\oplus(n+1)} \bigoplus_{s=1}^{\oplus(n+1)} (\mathbb{Z}/p^{n+1})^s / \langle \pm 1 \rangle$.

$(C_{p^n} \xrightarrow{\text{triv}} O(1)) \xrightarrow{\quad} ((1, 0, \dots, 0), (1, \dots, 1))$

$(C_{p^n} \xrightarrow{\quad} O(2)) \xrightarrow{\quad} ((2, 0, \dots, 0, -1, 0, \dots, 0), (1, \dots, 1, \overbrace{\frac{\delta}{v_p(j)}}^{0^{\text{th}}}, 1, \dots, 1))$

$\uparrow \text{generator} \quad \uparrow e^{2\pi i j/n}$

$\uparrow \text{p-valuation of } j \quad \uparrow \text{p-valuation of } j$

$\uparrow \text{0}^{\text{th}} \quad \uparrow \text{st} \quad \uparrow \text{0}^{\text{th}} \quad \uparrow \text{0}^{\text{th}}$

$\text{coprime-to-p part of } j$

$$\text{Cor. 1: } \text{Pic}(\text{DMack}_{C_{p^r}}) \cong \mathbb{Z}^{\oplus(n+1)} \oplus \left(\bigoplus_{s=1}^n (\mathbb{Z}/p^{n-s+1})^\times / \{\pm 1\} \right)$$

Cor. 3: For any $\overset{\Psi}{L} \leftrightarrow (\vec{\beta}, \vec{\gamma})$ (e.g. $L = \$^\vee$, using Cor. 2),

$$H_{C_{p^r}}^{i+L}(pt) \cong \left(G/H \longmapsto H_{-i} \left(\text{holim} \left(\begin{array}{c} \Sigma^{\beta_0} Z_{n-a}^a \\ \downarrow g_{n-a}^{a-1} \\ \Sigma^{\beta \leq 1} C_{n-a}^{a-1} \\ \downarrow g_{n-a}^{a-2} \\ \vdots \\ \Sigma^{\beta \leq 1} T_{n-a}^{a-2} \\ \downarrow g_{n-a}^0 \\ \vdots \\ \Sigma^{\beta \leq (a-1)} T_{n-a}^0 \end{array} \right) \right) \right)$$

where:

(a) We define the object

$$Z_r^a := \left(\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \underbrace{\mathbb{Z}[C_{p^r}] \xrightarrow{1-\sigma} \mathbb{Z}[C_{p^r}] \xrightarrow{p^a N} \mathbb{Z}[C_{p^r}] \xrightarrow{1-\sigma} \mathbb{Z}[C_{p^r}] \xrightarrow{p^a N} \cdots} \right).$$

(b) For any $i \geq 0$, we define the object

$$C_r^i := \left(\cdots \xrightarrow{p^{i+1}N} \mathbb{Z}[C_{p^r}] \xrightarrow{1-\sigma} \mathbb{Z}[C_{p^r}] \xrightarrow{p^{i+1}N} \underbrace{\mathbb{Z}[C_{p^r}] \xrightarrow{1-\sigma} \mathbb{Z}[C_{p^r}] \xrightarrow{p^i N} \mathbb{Z}[C_{p^r}] \xrightarrow{1-\sigma} \mathbb{Z}[C_{p^r}] \xrightarrow{p^i N} \cdots} \right).$$

(c) For any $i \geq 0$, we define the object

$$T_r^i := \left(\cdots \xrightarrow{p^{i+1}N} \mathbb{Z}[C_{p^r}] \xrightarrow{1-\sigma} \mathbb{Z}[C_{p^r}] \xrightarrow{p^{i+1}N} \underbrace{\mathbb{Z}[C_{p^r}] \xrightarrow{1-\sigma} \mathbb{Z}[C_{p^r}] \xrightarrow{p^{i+1}N} \mathbb{Z}[C_{p^r}] \xrightarrow{1-\sigma} \cdots} \right).$$

Q's: Multiplicative structure? Other groups? Other G -spaces?

Thanks for listening!