

# Reflection

## in Geometry Algebra & ~~Topology~~

Memoirs of the AMS (to appear)

from "Stratified noncommutative geometry", ~~arXiv 1910.14602~~

joint with David Ayala & Nick Rozenblyum

these slides: [etale.site/writing/caltech-reflection.pdf](http://etale.site/writing/caltech-reflection.pdf)

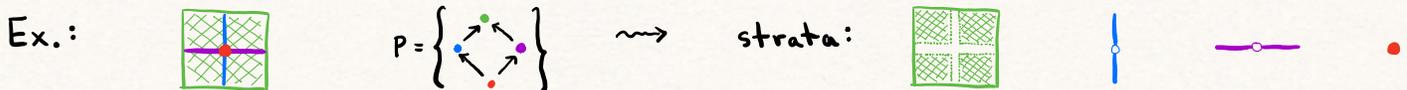
### §0: Overview

Notation:  $\hookrightarrow \text{St} :=$  stable  $\infty$ -cats  $\approx$  triangulated cats [e.g. dgVect, QCoh, Shv, coShv, ...]  
[ignore small vs. presentable]  
 $\hookrightarrow P$  a down-finite poset  $[\forall p \in P, (\leq_p) := \{q \in P : q \leq p\}$  is finite]

General idea: Given  $X \in \text{St}$  with P-stratification,  
 reconstruct  $X$  from: to be defined! [e.g. QCoh(strat<sup>d</sup> scheme), Shv(strat<sup>d</sup> space)]  
( $\approx$  poset-indexed filtration)

$\hookrightarrow$  its strata  $\{X_p \in \text{St}\}_{p \in P}$  "subquotients of a P-filtration" [e.g. Shv(strata of strat<sup>d</sup> space)]

$\hookrightarrow$  gluing data between strata.

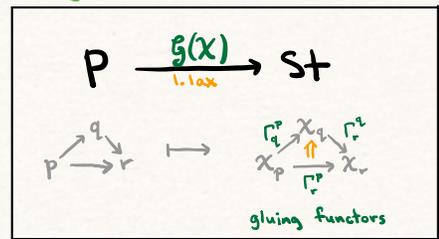
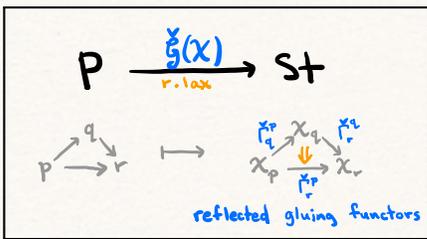


MAIN THM: Strata assemble in two ways, and can reconstruct  $X$  from both:

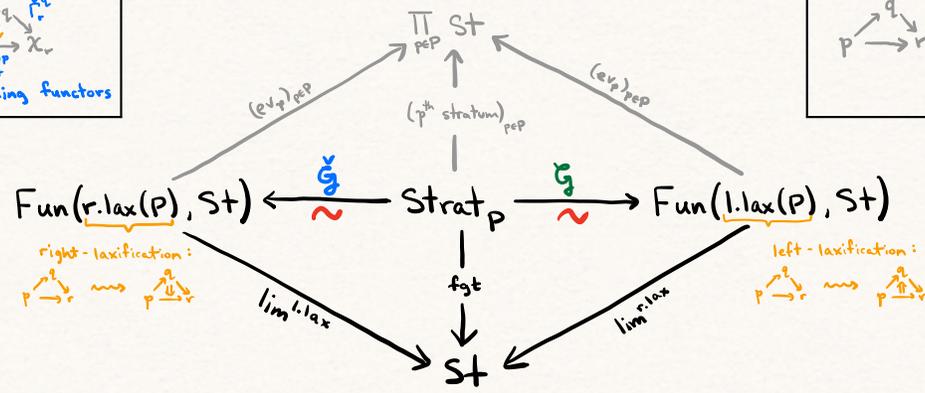
$$\lim^{l.lax} (\mathcal{G}(X)) \xleftarrow{\sim} X \xrightarrow{\sim} \lim^{r.lax} (\mathcal{G}(X))$$

reflected gluing diagram

gluing diagram



I.e.:



In particular, reflection:  $Fun(r.lax(P), St) \xleftarrow{\sim} Fun(l.lax(P), St)$

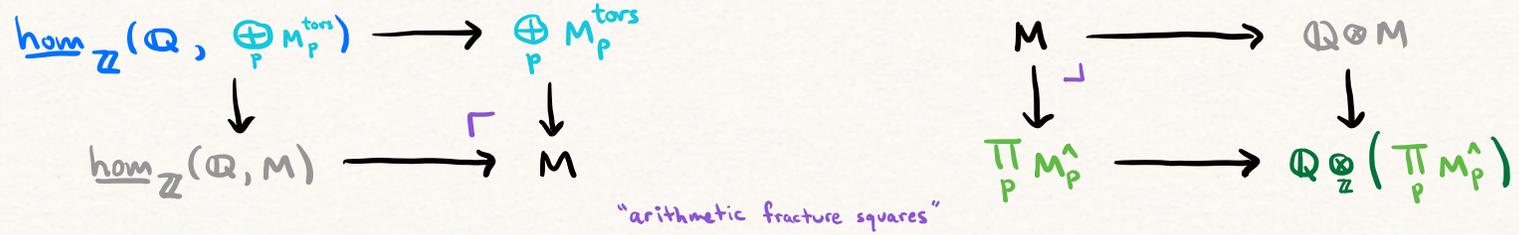
Explicit formula: for  $p < q$ ,

$$\check{\Gamma}_q^p \cong \text{tfib}_{\varphi \in \text{sd}(P)_{pq}^p} (\Sigma^{-1} \Gamma_\varphi) \quad \text{and} \quad \Gamma_q^p \cong \text{tcofib}_{\varphi \in (\text{sd}(P)_{pq}^p)^{op}} (\Sigma \check{\Gamma}_\varphi)$$

poset of sequences  $\{p=r_0 < r_1 < \dots < r_n=q\} =: \varphi$   
 $\hookrightarrow$  initial obj.  $\{p < q\}$   $\rightsquigarrow \Gamma_\varphi = \Gamma_{r_n}^{r_{n-1}} \dots \Gamma_{r_1}^{r_0}$

total co/fiber := failure to be a co/limit diagram  
 has final obj.  $\{p < q\}^o$

Ex.:  $M \in \text{Mod}_{\mathbb{Z}}$  can be reconstructed as:



Formula:  $\Sigma(\text{hom}_{\mathbb{Z}}(\mathbb{Q}, \bigoplus_P M_P^{\text{tors}})) \cong \mathbb{Q} \otimes_{\mathbb{Z}} (\prod_P M_P^{\wedge})$

Ex. (more later):  $T$  a  $P$ -strat<sup>d</sup> space w/ strata  $T_p \xrightarrow{\sigma_p} T$   
 i.e. cts fcn  $T \xrightarrow{f} P$   $T_p \xrightarrow{\sigma_p} T$   
locally closed  
 $f^{-1}(p)$

$\rightsquigarrow \mathcal{X} := \text{Shv}(T)$  is  $P^{\text{op}}$ -stratified, w/ strata  $\mathcal{X}_p = \text{Shv}(T_p)$ ;  
 for  $p \rightarrow q$  in  $P^{\text{op}}$ ,  $\mathcal{X}_p \xrightarrow{\Gamma_q^p} \mathcal{X}_q$   
 i.e.  $T_p \cong T_q$   $\text{Shv}(T_p) \xrightarrow{(\sigma_q^! \sigma_p)_!} \text{Shv}(T_q)$

§ 1: Examples & applications

§ 2: Stratified categories

# §1: Examples & applications

Background (lax limits): For  $I \xrightarrow{c} \text{Cat}$ ,

$$\lim(c) \simeq \left\{ \text{objects } (X_i \in \mathcal{C}_i)_{i \in I}, \text{ iso's } (X_i \xrightarrow{c_q} c_q(X_i) \cong X_j)_{i, j \in I}, \dots \right\}$$

$$\lim^{l.lax}(c) \simeq \left\{ \text{objects } (X_i \in \mathcal{C}_i)_{i \in I}, \text{ mor's } (X_i \xrightarrow{c_q} c_q(X_i) \xrightarrow{\gamma_q} X_j)_{i, j \in I}, \dots \right\}$$

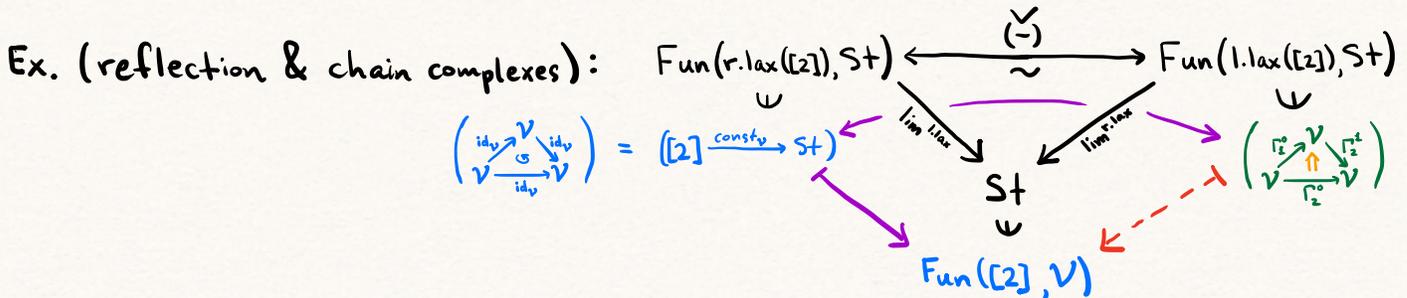
$$\lim^{r.lax}(c) \simeq \left\{ \text{objects } (X_i \in \mathcal{C}_i)_{i \in I}, \text{ mor's } (X_i \xrightarrow{c_q} c_q(X_i) \xleftarrow{\gamma_q} X_j)_{i, j \in I}, \dots \right\}$$

E.g. for  $I \xrightarrow{\text{const}_c} \text{Cat}$ ,  $\lim^{l.lax} \simeq \text{Fun}(I, \mathcal{C})$  and  $\lim^{r.lax} \simeq \text{Fun}(I^{\text{op}}, \mathcal{C})$ .

Generalization: e.g.

$$\lim^{r.lax} \left( \begin{array}{ccc} & B & \\ F \nearrow & & \searrow G \\ A & \xrightarrow{H} & C \end{array} \right) \simeq \left\{ \left( A, \begin{array}{ccc} B & & \\ \downarrow \gamma_1 & & \\ C & \xrightarrow{\gamma_2} & GB \\ \downarrow \gamma_2 & \circlearrowleft & \downarrow G(\gamma_1) \\ HA & \xrightarrow{\alpha_A} & GFA \end{array} \right) \right\}$$

(right-lax lim of left-lax diagram)



Recall formula: for  $p < q$  in  $P$ ,  $\Gamma_q^p \simeq \text{tcofib}_{\varphi^0 \in (\text{sd}(P))_1^p}(\Sigma \check{\Gamma}_q^p)$ .  
total cofiber := failure to be a colimit diagram  
poset of  $\{p_0, \dots, p_n = q\}$ 's

$$\Gamma_1^0 \simeq \text{tcofib}(\Sigma \check{\Gamma}_1^0) \simeq \Sigma, \quad \Gamma_2^1 \simeq \text{tcofib}(\Sigma \check{\Gamma}_2^1) \simeq \Sigma,$$

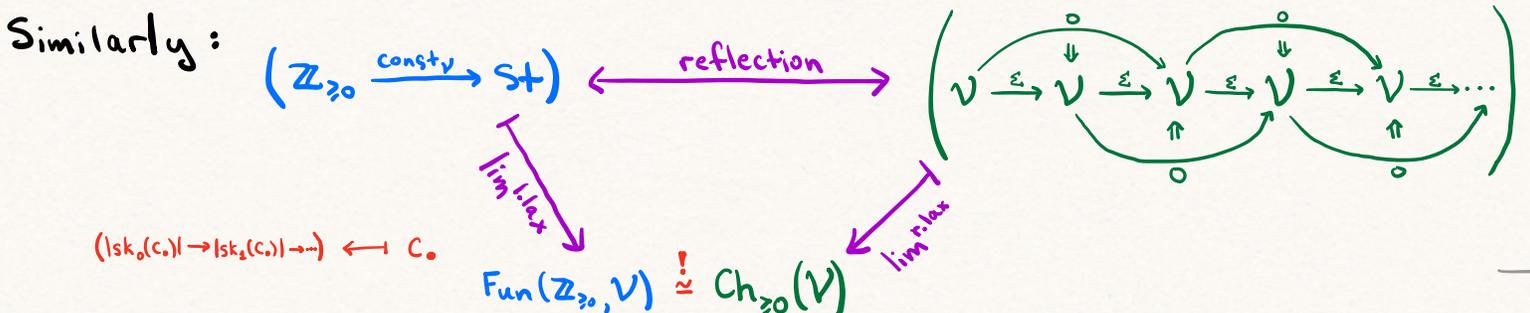
$$\Gamma_2^0 \simeq \text{tcofib}(\Sigma \check{\Gamma}_2^1 \check{\Gamma}_1^0 \rightarrow \Sigma \check{\Gamma}_2^0) \simeq \text{cofib}(\Sigma \xrightarrow{\sim} \Sigma) \simeq 0$$

failure = 0: it is a colimit diagram!

Hence:

$$\text{Fun}([2], V) \simeq \lim^{r.lax} \left( \begin{array}{ccc} \varepsilon \nearrow & V & \searrow \varepsilon \\ V & \xrightarrow{G} & V \\ \downarrow \varepsilon & \circlearrowleft & \downarrow \varepsilon \end{array} \right) \simeq \left\{ \left( \begin{array}{ccc} v_1 & & \\ \downarrow \varepsilon_1 & & \\ v_2 & \xrightarrow{G} & \varepsilon v_1 \\ \downarrow \varepsilon_2 & \circlearrowleft & \downarrow \varepsilon_1 \\ 0 & \xrightarrow{G} & \varepsilon v_2 \end{array} \right) \right\} \simeq \text{Ch}_{[2]}(V)$$

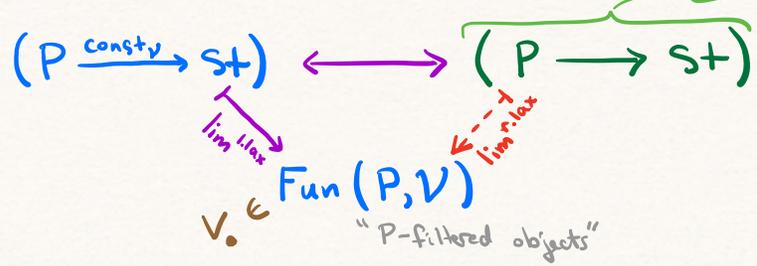
3-term chain complexes



Background: Given  $P \xrightarrow{f} A \in \text{Ab}$ , define  $P \xrightarrow{\hat{f}} A$  by  $\hat{f}(p) = \sum_{q \leq p} f(q)$ . Fourier transform of  $f$

Prop. (Möbius inversion):  $f(p) = \sum_{q \leq p} \underbrace{\mu(q, p)}_{\text{Möbius function}} \cdot \underbrace{\hat{f}(q)}_{\text{down-closure basis}}$ .  
Fourier inversion pointwise basis

Ex. (categorified Möbius inversion):



all values  $\mathcal{V}$ ;  
 for  $q \leq p$ ,  $\mathcal{V} \xrightarrow{\Gamma_p^q} \mathcal{V}$   
 $\mathcal{V} \xrightarrow{\quad} M_p^q \otimes \mathcal{V}$   
 $M_p^q := \sum_{\{q, p\}} |P_{q/p} \setminus \{q, p\}| \in \text{Spaces}_{\text{fin}}$   
 Möbius space

$\text{fil}_p(V_0) := V_p$  ,  $\text{gr}_p(V_0) := \text{tcofib}_{(\leq p)}(V_0)$

Cor.: Get filtration of  $\text{gr}_p(V_0)$  over  $(\leq p)^{\text{op}}$ , with  $\text{gr}_q = \Gamma_p^q(V_0) \simeq M_p^q \otimes \text{fil}_q(V_0)$ .  
pointwise basis down-closure basis

~> Möbius inversion in  $K_0(\mathcal{X})$

[Phillip Hall's Thm:  $\bar{\chi}(M_p^q) = \mu(q, p)$ .]  
 reduced Euler char.

Rmk.: get Möbius spectral sequence

$P \xrightarrow{d} \mathbb{Z}$   
 $E_{s,t}^1 = \bigoplus_{\substack{d(q)=s \\ q \leq p}} \pi_{s+t}(M_p^q \otimes \text{fil}_q(V_0)) \implies \pi_{s+t}(\text{gr}_p(V_0))$

e.g. for  $T$  a  $P$ -strat<sup>d</sup> space,

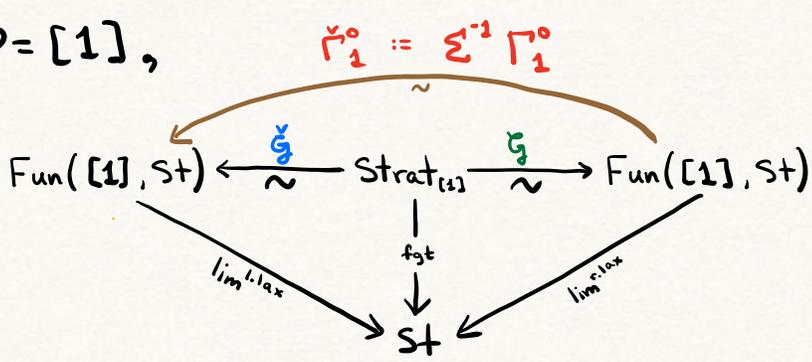
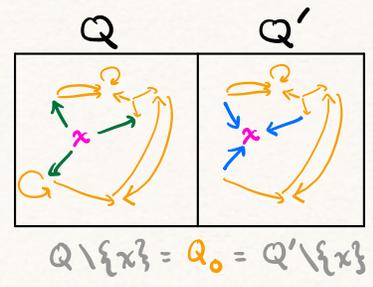
after Petersen  $E_{s,t}^1 = \bigoplus_{\substack{d(q)=s \\ q \leq p}} H_c^s(\bar{T}_q; \tilde{H}^t(M_p^q)) \implies H_c^{s+t}(T_p)$ .  
closure of q<sup>th</sup> stratum p<sup>th</sup> stratum

Ex. (BGP reflection):  $Q$  a quiver,  $x \in Q$  a source,  $Q'$  := mutation at  $x$   
 after Dyckerhoff-Jasso-Walde

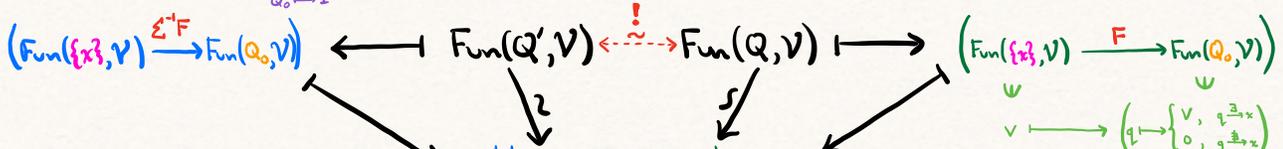
Thm.:  $\text{Fun}(Q', \mathcal{V}) \simeq \text{Fun}(Q, \mathcal{V})$ .

Pf.: For  $P = [1]$ ,

w/ finitely many adjacent edges



Functors  $(Q')^{\text{op}} \xrightarrow{x \mapsto 0^0, Q_0 \mapsto 1^0} [1]^{\text{op}}$  and  $Q \xrightarrow{x \mapsto 0, Q_0 \mapsto 1} [1]$  give strat<sup>ns</sup>; compute:



$$\lim_{[1]}^{l.lax} (\Sigma^{-1}F) \simeq \lim_{[1]}^{r.lax} (F)$$

Cor.:  $Q, Q'$  any orientations of same underlying finite tree  $\rightsquigarrow \text{Fun}(Q, \mathcal{V}) \simeq \text{Fun}(Q', \mathcal{V})$ .

# §2: Stratified categories

## §2.1: Stratifications over $P=[1]$

Given

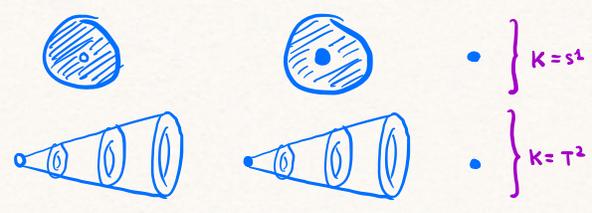
$$U \xrightarrow[\text{open}]{j} X \xleftarrow[\text{closed}]{i} Z$$

get

$$\text{Shv}^{l.c.}(U) \xleftarrow[\perp]{j^!} \text{Shv}^{cbl}(X) \xrightarrow[\perp]{i_*} \text{Shv}^{l.c.}(Z)$$

$l.c. :=$  locally constant  
 $cbl :=$  constructible (w.r.t. this closed open decomp.)

$$K \times (0,1] \xrightarrow[\text{open}]{\text{cone on } K \in \text{Top}} C(K) \xleftarrow[\text{closed}]{\text{pt}}$$



$$\begin{array}{ccccc} \text{Shv}^{l.c.}(K \times (0,1]) & \xleftarrow[\perp]{\text{!-stalk}} & \text{Shv}^{cbl}(C(K)) & \xleftarrow[\perp]{\text{* -stalk}} & \text{Shv}^{l.c.}(\text{pt}) \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ \text{Shv}^{l.c.}(K) & \xleftarrow[\perp]{d^*} & \text{Fun}(\text{Exit}(C(K)), \mathcal{V}) & \xleftarrow[\perp]{!-stalk} & \mathcal{V} \\ \cong \downarrow & & \cong \downarrow & & \downarrow \\ \text{Fun}(\Pi_{\infty}(K), \mathcal{V}) & & \text{Fun}(\Pi_{\infty}(K)^{\Delta}, \mathcal{V}) & & \end{array}$$

Abstract Def<sup>n</sup>: A recollement is a diagram  $\rightsquigarrow$  strat<sup>n</sup> of  $X$  over  $[1]$  s.t. all three sequences are exact.

$$\begin{array}{ccccc} K & \xrightarrow[\perp]{i_L} & X & \xrightarrow[\perp]{P_L} & Q \simeq X/K \\ \text{kernel} \swarrow & & \downarrow y & & \downarrow v \\ & & X & \xleftarrow[\perp]{i_R} & Q \\ & & \downarrow & & \downarrow p_R \\ & & & & \text{quotient} \end{array}$$

$$\text{im}(i_L) = \ker(p_L), \quad \text{im}(v) = \ker(y), \quad \text{im}(i_R) = \ker(p_R)$$

$y =$  restricted Yoneda  
 $v =$  null objects w.r.t.  $y$

ORDINARY

Microcosm Reconstruction Theorem:  $\forall F \in X$ , can reconstruct  $F$  as a pullback:

i.e. object-level

$$\begin{array}{ccc} F & \longrightarrow & v(p_L F) \\ \downarrow \perp & & \downarrow \delta_F \leftarrow \text{gluing map} \\ i_R(yF) & \xrightarrow{\eta} & v_{P_L i_R}(yF) \end{array}$$

...and this pullback square for  $F$  is unique:

ORDINARY

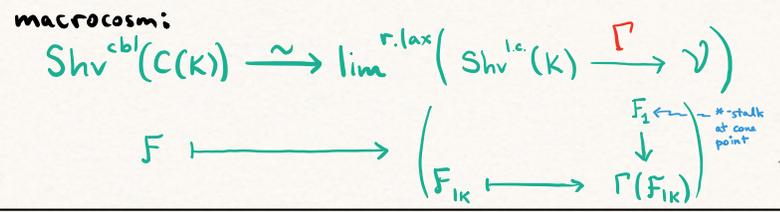
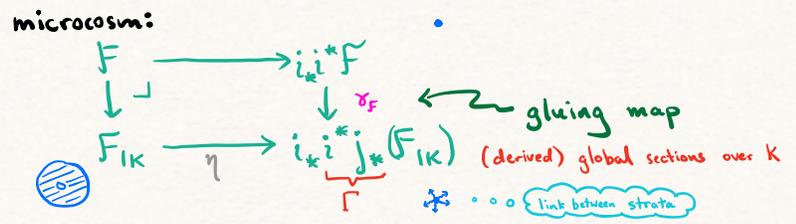
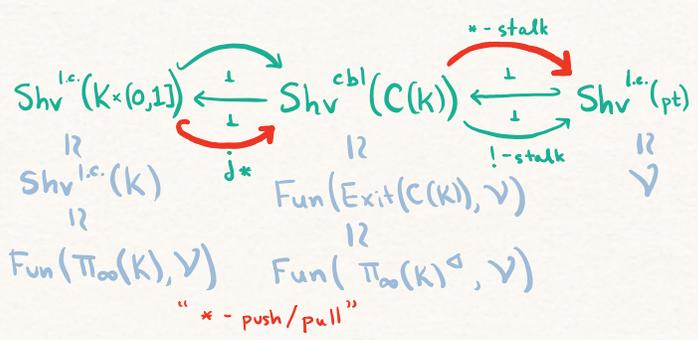
Macrocosm Reconstruction Theorem:

i.e. category-level

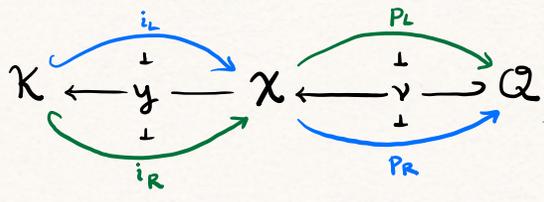
$$X \xrightarrow{\sim} \lim^{r.lax} (K \xrightarrow[\perp]{P_L i_R} Q)$$

$$F \longmapsto \left( \begin{array}{c} (P_L F) \\ \downarrow \delta_F \end{array} \right)$$

$$(yF) \mapsto \underline{P_L i_R}(yF)$$



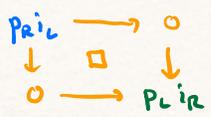
Return to recollement:



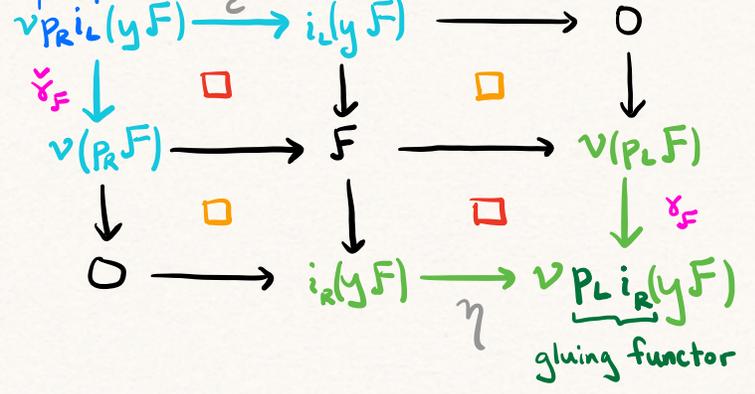
e.g.  
 $P_{Ri_L} = i^! j_!$   
 $P_{Li_R} = i^* j_*$

Microcosm Reconstruction:  $\forall F \in \mathcal{X}$ ,

...hence

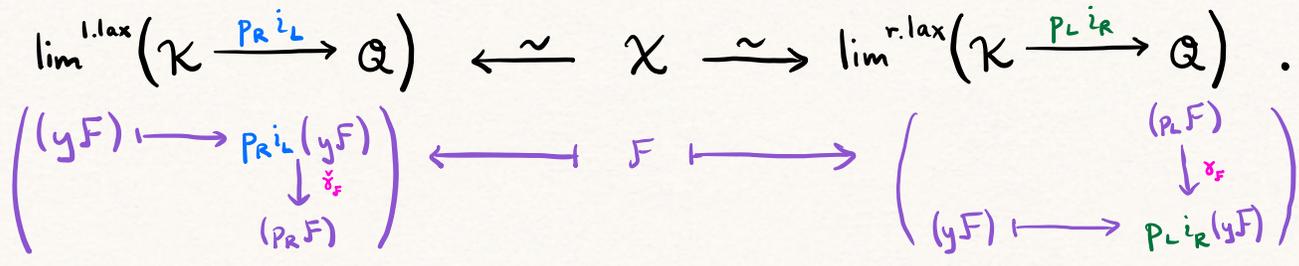


reflected gluing functor

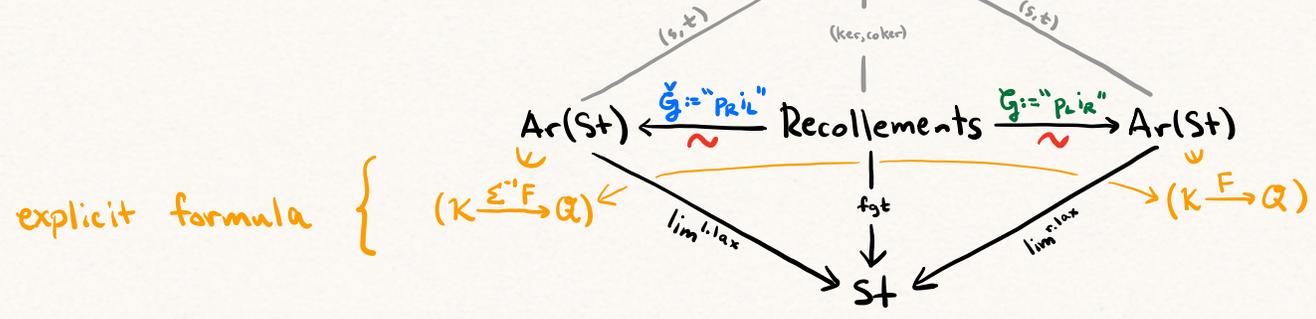


i.e.  $\varepsilon_{P_{Ri_L}}$  e.g.  $\varepsilon_{i^! j_! (F_{IK})}$  &  $\varepsilon_{\text{hom}_{\mathbb{Z}}(\mathbb{Q}, \oplus_{\mathbb{P}} M_{\mathbb{P}}^{\text{tors}})}$

Macrocosm Reconstruction:



Metacosm Reconstruction:



explicit formula {

# Reflection & Verdier duality

Obs.:  $\text{Rec} \xrightarrow[\sim]{(-)^\circ} \text{Rec}$

$$\left( \begin{array}{c} \begin{array}{ccc} K & \xleftarrow{i_L} & X \\ \perp & & \perp \\ \begin{array}{c} \leftarrow y \\ \leftarrow z \end{array} & & \\ \perp & & \perp \\ K & \xrightarrow{i_R} & X \end{array} & \xrightarrow{\begin{array}{c} P_L \\ \perp \\ \perp \\ \perp \\ P_R \end{array}} & \begin{array}{ccc} K^* & \xleftarrow{i_R^*} & X^* \\ \perp & & \perp \\ \begin{array}{c} \leftarrow y^* \\ \leftarrow z^* \end{array} & & \\ \perp & & \perp \\ K^* & \xrightarrow{i_L^*} & X^* \end{array} & \xrightarrow{\begin{array}{c} P_R^* \\ \perp \\ \perp \\ \perp \\ P_L^* \end{array}} & \begin{array}{ccc} K^* & \xleftarrow{i_R^*} & X^* \\ \perp & & \perp \\ \begin{array}{c} \leftarrow y^* \\ \leftarrow z^* \end{array} & & \\ \perp & & \perp \\ K^* & \xrightarrow{i_L^*} & X^* \end{array} \end{array} \right) \rightsquigarrow \check{G}(X) = (G(X^\circ))^\circ$$

Verdier duality: for  $T \in \text{Top}^{\text{LCH}}$ ,  $\text{Shv}_V(T)^\circ \xleftarrow[\sim]{\mathbb{D}} \text{Shv}_{V^{\text{op}}}(T)$ .

[Lurie]  $\mathcal{F} \longmapsto (u \mapsto \Gamma_c(u; \mathcal{F}))$

Rmk.: usual Verdier duality:  $\text{Hom}_{\text{Shv}_V(T)}(-, \omega_T)$

$\omega_T := p^! \mathbb{1}$  for  $T \rightarrow \text{pt}$  (e.g. shifted orientation sheaf of a manifold)

postcomp. with  $\text{hom}_V(i; \mathbb{1})$  i.e. "linear dual" in  $\mathcal{V}$

Key feature: For  $U \xrightarrow[\text{open}]{j} T$ ,

$$\begin{array}{ccc} \text{Shv}_V(T)^\circ & \xleftarrow[\sim]{\mathbb{D}} & \text{Shv}_{V^{\text{op}}}(T) \\ \uparrow (j_*)^\circ & & \uparrow j_* \\ \text{Shv}_V(U)^\circ & \xleftarrow[\sim]{\mathbb{D}} & \text{Shv}_{V^{\text{op}}}(U) \end{array}$$

Cor.: Verdier duality interchanges ordinary and reflected reconstruction.

(More generally:  $T \xrightarrow{f} P \rightsquigarrow$  in  $\text{Strat}_{P^{\text{op}}}$ ,  $\text{Shv}_V(T) \xrightarrow[\text{"reflection"}]{\sim} \text{Shv}_V(T)^\circ \xleftarrow[\sim]{\mathbb{D}} \text{Shv}_{V^{\text{op}}}(T)$ .)

Rmk.: For e.g.  $T = (\mathbb{R}^0 \subset \mathbb{R}^1)$ ,

$$\begin{array}{ccc} \text{Shv}_V^{P=0, S1}(T)^\circ & \xleftarrow[\sim]{\mathbb{D}} & \text{Shv}_{V^{\text{op}}}^{P=0, S1}(T) \\ \text{Fun}(\bullet \leftarrow \bullet \rightarrow \bullet, \mathcal{V})^\circ & \xleftarrow[\sim]{\text{BGP reflection}} & \text{Fun}(\bullet \leftarrow \bullet \rightarrow \bullet, \mathcal{V})^\circ \end{array}$$

## §2.2: Stratifications over P (I.O.U. definitions)

$X \in \text{St}$ ,  $\text{Ker}_X := \{ K \xrightarrow{i} X \}$

poset of kernels of recollements of  $X$  (normal subgroups)

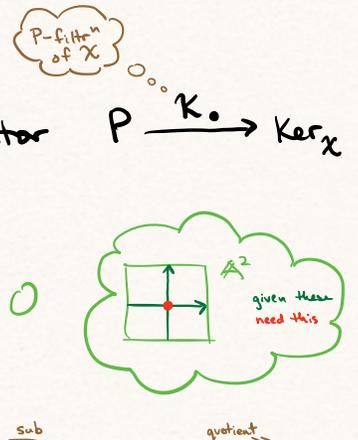
Def<sup>n</sup>: A stratification of  $X$  over  $P$  is a functor  $P \xrightarrow{K_\bullet} \text{Ker}_X$  st.:

① generation:  $\bigcup_{p \in P} K_p = X$  ;

② stratification:  $\forall p, q \in P$ ,  $\bigcup_{\text{resp } \text{resq}} K_r \xrightarrow{\quad} K_p$

$\exists i: K_q \xrightarrow{\quad} K_p$

$\exists i: K_q \xrightarrow{\quad} X$



The  $p^{\text{th}}$  stratum is  $X_p := K_p / K_{(c,p)}$ , a subquotient:  $X \begin{matrix} \xleftarrow{i_L} \\ \xrightarrow{i_R} \end{matrix} K_p \begin{matrix} \xrightarrow{p_L} \\ \xleftarrow{p_R} \end{matrix} X_p$   
 $\simeq \text{coFib}_{(c,p)}(K_c) \cup_{q=p} K_q$

	REFLECTED	ORDINARY
localization adj's		
gluing fctrs for p to q in P	$\Gamma_q^P: X_p \xleftarrow{\lambda^P} X \xrightarrow{\Phi_q} X_q$	$\Gamma_q^P: X_p \xleftarrow{\rho^P} X \xrightarrow{\Phi_q} X_q$
gluing diagram	$\check{G}: P \xrightarrow{\text{r.lax}} \text{St}$ 	$G: P \xrightarrow{\text{l.lax}} \text{St}$ 

$P = [2]$ , ordinary reconstruction:

Macrocosm:

$$X \xleftarrow[\text{regluing}]{\sim} \lim_{\text{r.lax}} \left( \begin{array}{ccc} & X_1 & \\ \Gamma_1^0 \nearrow & \uparrow \eta & \searrow \Gamma_2^1 \\ X_0 & & X_2 \\ & \Gamma_2^0 \searrow & \end{array} \right) := \left\{ \left( \begin{array}{c} \boxed{F_1} \\ \downarrow \gamma_{01} \\ \boxed{F_0} \end{array} \right), \left( \begin{array}{ccc} \boxed{F_2} & \xrightarrow{\gamma_{12}} & \Gamma_2^1 \boxed{F_1} \\ \gamma_{02} \downarrow & \gamma_{012} \curvearrowright & \downarrow \Gamma_2^1 \gamma_{01} \\ \Gamma_2^0 \boxed{F_0} & \xrightarrow{\gamma} & \Gamma_2^1 \Gamma_1^0 \boxed{F_0} \end{array} \right) \right\}$$

Microcosm:

$$X \ni \lim \left( \begin{array}{ccc} \boxed{2} & \xrightarrow{\quad} & \boxed{12} \\ \downarrow \gamma_{02} & \nearrow \gamma & \downarrow \gamma_{012} \\ \boxed{02} & \xrightarrow{\quad} & \boxed{012} \\ \uparrow \gamma & \downarrow \gamma & \\ \boxed{0} & \xrightarrow{\quad} & \boxed{01} \end{array} \right)$$

$\text{sd}([2])$

"include back into  $X$  via  $X_i \xleftarrow{p^i} X$ "

Macrocosm Reconstruction ( $P$  arbitrary):

$$\text{Fun}(\text{sd}(P), X) \cong \lim_{\text{l.lax}} \left( P \xrightarrow[\text{r.lax}]{\check{G}(X)} \text{St} \right) \xleftarrow[\check{G}]{\text{colim}_{\text{sd}(P)} \sim} X \xleftarrow[\lim_{\text{sd}(P)}]{\sim} \lim_{\text{r.lax}} \left( P \xrightarrow[\text{l.lax}]{\check{G}(X)} \text{St} \right) \subseteq \text{Fun}(\text{sd}(P), X)$$

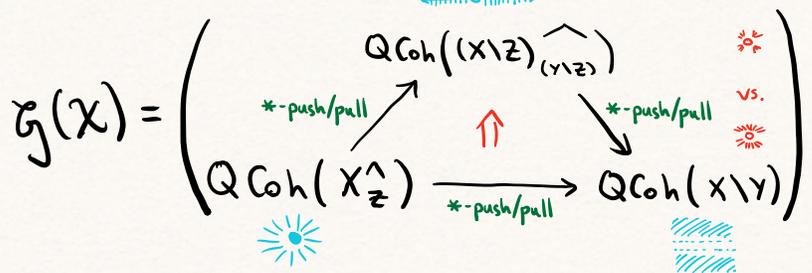
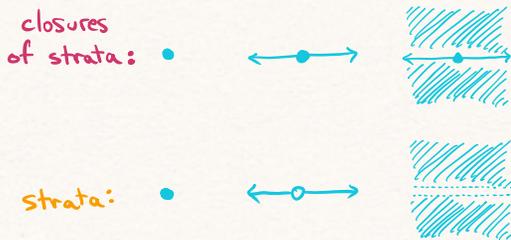
the subdivision of  $P$ :  $\{ob = \{p_0, \dots, p_n\}, mor = inclusion\}$   
 e.g.  $\text{sd}(\mathbb{N}) = \text{punctured } (\mathbb{N}+2)\text{-cube}$

$$\left( ([n] \xrightarrow{\varphi} P) \mapsto \prod_{\varphi} \Phi_{\varphi(0)}(F) \in \chi_{\varphi(n)} \xrightarrow{\lambda^{\text{in}}} X \right) =: \check{g}(F) \longleftarrow F \longleftarrow g(F) := \left( ([n] \xrightarrow{\varphi} P) \mapsto \prod_{\varphi} \Phi_{\varphi(0)}(F) \in \chi_{\varphi(n)} \xrightarrow{\rho^{\text{in}}} X \right)$$

example	strata	$\Gamma$	$\vec{\Gamma}$
1) Qcoh of strat <sup>d</sup> scheme $P \xrightarrow{\mathbb{Z}_0} \text{Closed}_X \xrightarrow{\text{QCoh}_X(X)} \text{Ker}_{\text{Qcoh}(X)}$ <p>e.g. <math>(X, \text{Spec}^{\text{ad}}) \xrightarrow{\Gamma} \text{adelic strat}^n</math></p>	$\text{Qcoh}((U_p)_{X_p}^{\wedge})$	*-push/pull	local cohomology
2) Shv on strat <sup>d</sup> top spc $P^{\text{op}} \xrightarrow{U} \text{Open}_T \xrightarrow{(\text{Shv}, j_i)} \text{Ker}_{\text{Shv}(T)}$	$\text{Shv}(T_p)$	*-push/pull	!-push/pull
3) $\text{Fun}(T, \mathcal{V})$ for $\text{Cat}_0 \ni T \rightarrow P$ $P^{\text{op}} \xrightarrow{(\text{Fun}(T_0, \mathcal{V}), \text{LKE})} \text{Ker}_{\text{Fun}(T, \mathcal{V})}$ <p>e.g. <math>\text{Fun}(\mathbb{Z}_{30}, \mathcal{V})</math></p>	$\text{Fun}(T_p, \mathcal{V})$	*-push/pull along links	!-push/pull along links
4) genuine $G$ -spectra $P_G = \{\text{closed subgps, subconjugacy}\}$	$N(H)/H$ homotopy $W(H)$ -spectra	~Tate cohomology	[no name]
5) Spectra, chromatic strat <sup>n</sup> $P_{\text{Spectra}} = \left( \begin{array}{c} \text{0} \\ \text{1} \\ \text{2} \\ \text{3} \\ \text{4} \\ \text{5} \\ \dots \end{array} \right) = \text{Spec}^{\text{Balmer}}(\text{Spectra})$	$K(n)$ -local spectra	$K(n)$ -localization	$n^{\text{th}}$ monochromatic layer

e.g.  $[2]$ -strat<sup>d</sup> scheme:  $[2] \xrightarrow{\mathbb{Z} \hookrightarrow Y \hookrightarrow X} \text{Closed}_X \xrightarrow{\text{QCoh}_X(X)} \text{Ker}_{\text{Qcoh}(X)}$   
Set-theoretic support

sub-e.g.  $A^0 \subset A^1 \subset A^2$



Thanks for listening!



