

Towards
1

Knot homology for 3-manifolds

forthcoming joint work with:



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§ 1 : Knot polynomials

§ 2 : Knot homologies

§ 3 : Main theorem

* knot := link := "knot or link"

* ignore orientations & gradings

* q always generic

§ 1.1 : Knot polynomials for $L^1 \subset \mathbb{R}^3$

['84] Jones polynomial of $L^1 \subset \mathbb{R}^3$, $J_L(q) \in \mathbb{Z}[q^{\pm}]$

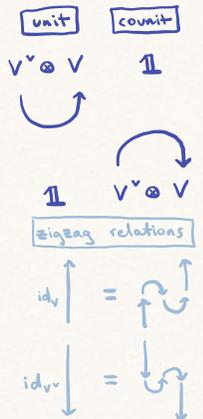
characterized by:

↳ normalization: $J(q) = 1$

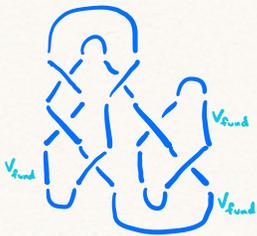
↳ skein relation: $(q - q^{-1}) \cdot J_{\text{unknot}}(q) = q^2 \cdot J_{\text{unknot}}(q) - q^{-2} \cdot J_{\text{unknot}}(q)$

[89] Witten: Jones poly. from Chern-Simons theory (QFT)

[90] Reshetikhin-Turaev made this rigorous using quantum groups, namely the braided(-monoidal) category $\mathcal{C} := \text{Rep}^{\text{f.d.}}(U_q(\mathfrak{sl}_2)) \ni V_{\text{fund.}}$:
 quantum \mathfrak{sl}_2 dualizable!!



present $L^2 \subset \mathbb{R}^3$ as a "string diagram", label by $V_{\text{fund.}}$.



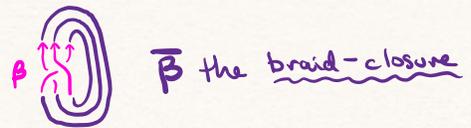
$J_L(q) \in \text{end}_{\mathcal{C}}(\mathbb{1}) \cong \mathbb{C}(q)$ (ground field)

Alternatively,

$$\text{Br}_n \xrightarrow[\beta]{f} \text{end}_{\mathcal{C}}(V_{\text{fund.}}^{\otimes n}) \xrightarrow[\text{trace}]{\text{using dualizability of } V_{\text{fund.}}} \text{end}_{\mathcal{C}}(\mathbb{1}) \cong \mathbb{C}(q)$$

$$\beta \longmapsto \text{tr}(f(\beta)) = J_{\bar{\beta}}(q)$$

Thm. (Alexander '23, Markov '36): $\frac{\{\text{braids}\}}{\sim} \xrightarrow{\text{closure}} \{\text{links in } \mathbb{R}^3\}$.



★ gen^{zns}: \mathfrak{sl}_N -poly. (e.g. Jones poly (N=2)) $\xrightarrow{\text{specialization of variables}}$ HOMFLY-PT poly. ("sl_∞") (two variables)

§1.2: Knot "polynomials" for $L^2 \subset M^3$ via skein modules

Given \mathcal{C} (e.g. $\text{Rep}^{\text{f.d.}}(U_q(\mathfrak{sl}_2))$) a braided K -linear cat. w/ duals ...

classical $sk_{\mathcal{C}}(M^3) := \mathbb{k} \left\{ \begin{array}{l} \text{links in } M \text{ labeled} \\ \text{by obj's of } \mathcal{C} \end{array} \right\} / \text{local relations}$
 use duality
 isotopy, skein relations, merge strands and tensor their labels
 use braiding

derived $sk_{\mathcal{C}}(M^3) := \int_{M^3} \mathcal{C} \in \mathcal{D}_{\geq 0}(\text{Mod}_{\mathbb{k}})$
 considered as a pointed \mathbb{k} -linear 3-cat. (2-fold delooping)
 factorization homology: $\int_{n\text{-fld}} \mathbb{V}\text{-enriched } n\text{-cat} \in \mathcal{V}$
 e.g. $sk_{\mathcal{C}}(\mathbb{R}^3) = \text{end}_{\mathcal{C}}(\mathbb{1})$
 $[n=1: \text{AMR '17}]$ $A = \text{Ayala}$
 $[n \geq 2: \text{AFMR, w.i.p.}]$ $F = \text{Francis}$
 $R = \text{Rozanbljum}$

$J_{(L^2 \subset M^3)}(\vec{V} \in \mathcal{C}) := \int_{(L^2 \subset M^3)} (\vec{V} \in \mathcal{C}) \forall \pi_0(L^2) \xrightarrow{\vec{V}} \text{obj}(\mathcal{C})$
 labeling of path components
 generalized Jones poly.

Toy version: invariants for $L^0 \in M^2$, given $A \in \text{Alg}_{\mathbb{k}} \dots$
 $A/[A,A] \in \text{Mod}_{\mathbb{k}}$
 considered as a pointed \mathbb{k} -linear 1-cat.

$\int_{\mathbb{R}^2} A = A$ $\int_{(L^0 \subset \mathbb{R}^2)} (\vec{a} \in A) = a_1 \cdot a_2 \cdot \dots \cdot a_k$ 	$\int_{S^1} A = \text{HH}(A) \in \mathcal{D}_{\geq 0}(\text{Mod}_{\mathbb{k}})$ $\int_{(L^0 \subset S^1)} (\vec{a} \in A) = [a_1 \cdot \dots \cdot a_k]$ 	e.g. for $A = \text{Mat}_{d \times d}(\mathbb{k})$, $\text{HH}(A) \cong \mathbb{k}$ $[a_1 \cdot \dots \cdot a_k] \leftrightarrow \text{tr}(a_1 \cdot \dots \cdot a_k)$ invariant under cyclic permutations isotopies of $L^0 \subset S^1$
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§2: Knot homologies

['99] Khovanov homology $\text{Kh}^{i,j}(L) (i, j \in \mathbb{Z})$, a categorification

of Jones poly.: $J_L(q) = \chi_{gr}(\text{Kh}(L)) := \sum_{i,j} (-1)^i \cdot \text{rk}(\text{Kh}^{i,j}) \cdot q^j$

* gen^{zns}: sl_N -homology (bigraded) ['04 Kh.-Rozansky]
 spectral sequence ['06 Rasmussen]
 HOMFLY-PT homology (triple-graded) ['05 Kh.-Roz., Kh., Dunfield-Gukov-Rasmussen]
 "sl..."

Major Q.: How to extend these to $L^1 \subset M^3$?

* approaches via physics: Witten ['11], Gaiotto-Witten ['11], Gukov-Pei-Ratrov-Vafa ['17], Gukov-Mandalescu ['19], Aganagic ['20], ...

Today's A.: seek braided \mathbb{k} -linear $(\infty, 2)$ -cat \mathcal{C} with duals
 cat_k-enr. 3-cat.

$\rightsquigarrow \text{Sk}_{\mathcal{C}}(M^3) := \left(\int_{M^3} \mathcal{C} \right) \ni \left(\int_{(L^2 \subset M^3)} (\vec{v} \in \mathcal{C}) \right) \leftarrow \text{gen}^{\text{rd}} \text{ knot hlg}$
 $\underbrace{\text{Sk}_{\mathcal{C}}(M^3)}_{\substack{\text{sk} \text{in} \\ \infty\text{-category}}} \quad \forall \pi_0(L^2) \xrightarrow{\vec{v}} \text{obj}(\mathcal{C})$

\star for sl_N -hlg, want $\mathcal{C} = \text{Rep}^{\text{f.d.}}(\mathcal{U}_q(sl_N))$
 e.g. Kh (N=2)

categorified quantum group: $\mathcal{U}_q(\mathfrak{g})$ a 2-cat
 [Chuang-Rouquier '04]
 [Lauda '08]
 [Khovanov-Lauda '08]
 ...

"2-rep^{ns}" := fctrs $\mathcal{U}_q(\mathfrak{g}) \rightarrow 2\text{Vect} := \text{Morita} := \begin{cases} \text{obj} = k\text{-alg's} \\ \text{hom}(A,B) = \text{Bim}_{(A,B)} \end{cases}$

Problem: No monoidal str. known - let alone braiding!

§3: Main theorem: braiding on " $\text{Rep}^{\text{f.d.}}(\mathcal{U}_q(sl_{\infty}))$ "

§3.1: Background on HHH := HOMFLY-PT homology

k a field of char. 0, $n \geq 0$

$\text{SBim}_n \subseteq \text{Bim}_{k[x_1, \dots, x_n]}$ the Soergel bimodules for S_n } additive k -linear
 } monoidal subcat.
 $\otimes := \otimes_{k[x_1, \dots, x_n]}$

$\mathcal{H}_n := K^b(\text{SBim}_n)$ the Hecke (∞ -)cat. for S_n } monoidal k -linear
 } stable ∞ -cat.
 $\hookrightarrow D_{\text{cl}}^b(B/G/B)$, monoidal by convolution

Thm (Rouquier '04): a monoidal fctr $\text{Br}_n \xrightarrow{F} \mathcal{H}_n$.

[Main Thm $\Rightarrow \mathcal{H}_n \cong \text{end}_{\mathcal{H}}(V_{\text{unk}}^{\otimes n})$ and F categorifies $\text{Br}_n \xrightarrow{f} \text{end}_{\mathcal{C}}(V_{\text{unk}}^{\otimes n})$]

Rmk.: A "deformation" of boring action: $\text{Br}_n \xrightarrow{F} \mathcal{H}_n$
 \times, \star
 $\downarrow \pi_{\infty}$
 "underlying weak stable htpg type" (ie. invert q, 's)

$$S_n \xrightarrow{\{x_1, \dots, x_n\} \hookrightarrow S_n} D^b(\text{Bim}_{K[x_1, \dots, x_n]}).$$

Ex.: For $\sigma = \text{crossing} \in Br_2$, $F(\sigma) := (R \otimes_{R^S} R \xrightarrow{\text{mult.}} R) \in K^b(\text{SBim}_2)$,
degree 0

boring! \times \rightarrow

$$H_i(F(\sigma)) \cong \begin{cases} K[x_1, x_2] \xrightarrow{e} K[x_1, x_2] \xrightarrow{s} K[x_1, x_2], & i=0 \\ K[x_1, x_2] \xrightarrow{e} 0 \xrightarrow{s} K[x_1, x_2], & i \neq 0 \end{cases}$$

$R := K[x_1, x_2] \hookrightarrow S_2 = \{e, s\}$
 U
 $R^S = K[x_1+x_2, x_1 \cdot x_2]$

Thm. (Kh. '05): $HHH(\bar{\beta}) := \mathbb{R} \text{hom}_{\mathcal{H}_n}(\mathbb{1}, F(\beta))$ a well-defined link invariant, categorifying HOMFLY-PT poly.
triple grading: gbins, ch cx., $R^i \text{hom}$ $F(e) = \mathbb{1}$

§ 3.2: Main theorem

Recall: $\mathcal{H}_n := K^b(\text{SBim}_n) \hookrightarrow K(\text{Bim}_{K[x_1, \dots, x_n]}) \xrightarrow{\Pi_\infty} D(\text{Bim}_{K[x_1, \dots, x_n]})$
"underlying weak stable htgy type"

Def.: $\mathcal{H} := \begin{cases} \text{ob} = \mathbb{N} \\ \text{hom}(m, n) = \begin{cases} \mathcal{H}_n, & m=n \\ 0, & m \neq n \end{cases} \end{cases}$ } a "stable K -linear $(\infty, 2)$ -cat"
"the Hecke 2-cat" (in type A) } (hom's are st. K -lin. $(\infty, 1)$ -cats)

Easy: $\mathcal{H}_i \times \mathcal{H}_j \xrightarrow{\boxtimes} \mathcal{H}_{i+j} \rightsquigarrow$ monoidal str. on \mathcal{H}
"parabolic induction"

s.t. $\Pi_\infty: \mathcal{H} \hookrightarrow K(\text{Morita}) \xrightarrow{\Pi_\infty} D(\text{Morita})$
 $\xrightarrow{n} K[x_1, \dots, x_n]$

$K[x_1, \dots, x_i] \otimes_K K[x_1, \dots, x_j] \cong K[x_1, \dots, x_{i+j}]$
 is monoidal.

$\begin{cases} \text{ob} = K\text{-alg's} \\ \text{hom}(A, B) = K(\text{Bim}_{(A, B)}) \\ \otimes := \otimes_K \end{cases}$ $\begin{cases} \text{ob} = K\text{-alg's} \\ \text{hom}(A, B) = D(\text{Bim}_{(A, B)}) \\ \otimes := \otimes_K \end{cases}$

∞ -categorical uniqueness: contractible ∞ -groupoid (but massive as a simplicial set!!)

Main Thm.: $\exists!$ braiding on \mathcal{H} s.t.:

Cor: naturality of Rouquier's Br_n -actions, e.g.

in $2\text{-hom}_{\mathcal{H}}(\mathbb{1}, \mathbb{1})$
id. homs

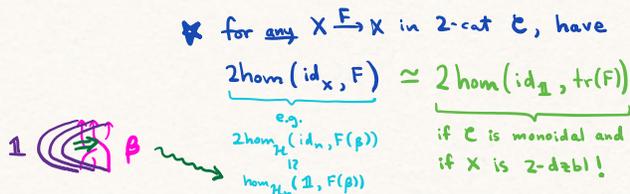
① braiding $(1 \boxtimes 1 \rightarrow 1 \boxtimes 1) \in \text{hom}_{\mathcal{H}}(2, 2) := \mathcal{H}_2$ is the Rouquier complex $F(\text{crossing}) := (R \otimes_{R^S} R \xrightarrow{\text{mult.}} R)$;

② $\mathcal{H} \xrightarrow{\Pi_\infty} \mathcal{D}(\text{Morita})$ is braided. " $\Pi_\infty(\text{braiding on } \mathcal{H}) = \text{braiding on } \mathcal{D}(\text{Morita})$ " (boring (symmetric) braiding)

Rmk.: Expect $\mathcal{H} \subset \text{Rep}(\mathcal{U}_q(\mathfrak{sl}_\infty))$ full braided sub- $(\infty, 2)$ -cat. on $V_{\text{fund.}} := 1 \in \mathcal{H}$. But \mathcal{H} (and its braiding) much easier to construct!

↳ Q.: Comparison via Tannakian reconstruction?

But still only for $L^1 \subset \mathbb{R}^3$: $V_{\text{fund.}} \in \mathcal{H}$ not 2-dualizable!
 Check: $2\text{end}_{\mathcal{H}}(V_{\text{fund.}}) := \text{hom}_{\mathcal{H}_1}(1, 1) = \mathbb{k}[x]$ (inf.-dim!)



2-dzblity \Rightarrow functoriality w.r.t. embedded cobordisms

So, work in progress: $\mathfrak{sl}_\infty \rightsquigarrow \mathfrak{sl}_N$

$\rightsquigarrow V_{\text{fund.}} \in \mathcal{H}_{\mathfrak{sl}_N}$ should be 2-dualizable

$\mathbb{k}[x] \rightsquigarrow \mathbb{k}[x]/x^{N+1}$ (inf.-dim! \rightsquigarrow f. dim!)

\rightsquigarrow hlg of knots in 3-mflds!!!

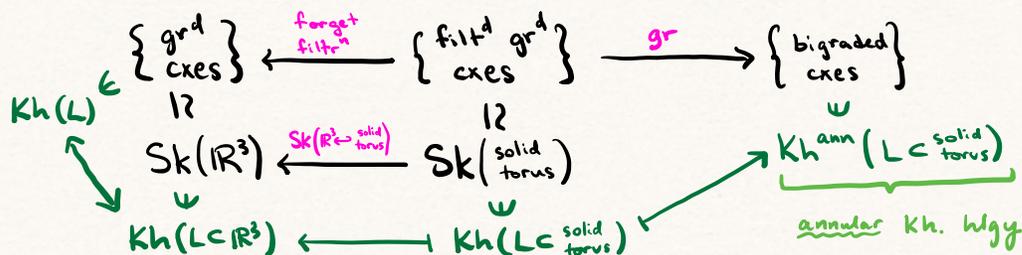
$\text{Cat}_{(\infty, 2)} \ni \text{Cob}^1(M^3) \xrightarrow{\mathfrak{sl}_N\text{-homology!}} \text{Sk}_{\mathcal{H}_{\mathfrak{sl}_N}}(M^3) := \int_{M^3} \mathcal{H}_{\mathfrak{sl}_N}$

- 0-mor's = links in M^3
- 1-mor's = link. cob's in $M^3 \times [0, 1]$
- 2-mor's = isotopies
- ...

$(L^1 \subset M^3) \longmapsto \int_{(L^1 \subset M^3)} (V_{\text{fund.}} \in \mathcal{H}_{\mathfrak{sl}_N})$

the \mathfrak{sl}_N skein category of M^3 (stable \mathbb{k} -linear (0,1)-)

E.g. for $N=2$, expect:

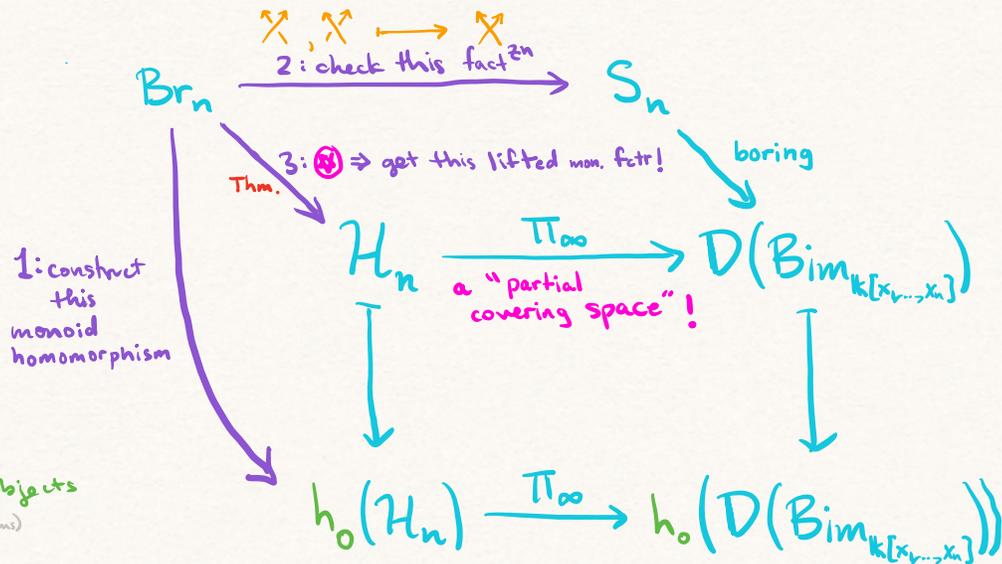


§3.3: Pf. idea

Main Thm.: $\exists!$ braiding on \mathcal{H} s.t.:

- ① braiding $(1 \otimes 1 \rightarrow 1 \otimes 1) \in \text{hom}_{\mathcal{H}}(2,2) := \mathcal{H}_2$
- ② $\mathcal{H} \xrightarrow{\pi_\infty} \mathcal{D}(\text{Morita})$ is braided.

Rouquier's Pf. idea:

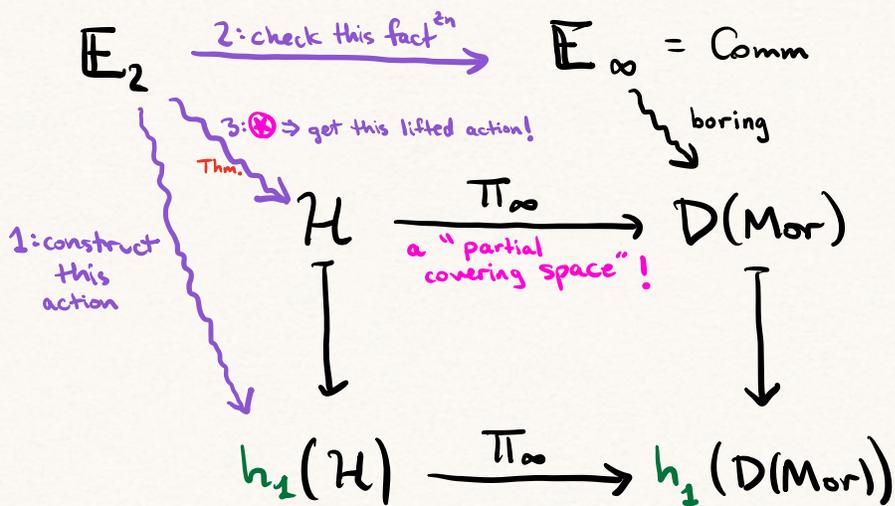


$h_0 :=$ set of isoclasses of objects
(ignore all noninvertible 1-morphisms)

I.e.: If $L, L' \in \mathcal{H}_n$ \otimes -invertible s.t. \exists equiv^{ce} $L \cong L'$,

$$\begin{array}{ccc} \text{hom}_{\mathcal{H}_n}(L, L') & \xrightarrow{\cong} & \text{hom}_{\mathcal{D}}(\pi_\infty(L), \pi_\infty(L')) \\ \text{SI} & & \text{SI} \\ \text{hom}_{\mathcal{H}_n}(\mathbb{1}, L^{-1} \otimes L') & \longrightarrow & \text{hom}_{\mathcal{D}}(\mathbb{1}, \pi_\infty(L)^{-1} \otimes \pi_\infty(L')) \\ \text{SI} & & \text{SI} \\ \text{hom}_{\mathcal{H}_n}(\mathbb{1}, \mathbb{1}) & \longrightarrow & \text{hom}_{\mathcal{D}}(\mathbb{1}, \mathbb{1}) \\ \text{SI} & & \text{SI} \\ \mathbb{K} & \longrightarrow & \mathbb{K} \end{array}$$

Pf. idea:



$h_2 :=$ homotopy $(1,1)$ -cat
(ignore all noninvertible 2-morphisms)

$$\rightsquigarrow \text{hom}_{h_2(\mathcal{H})}(n,n) := h_0(\mathcal{H}_n)$$

Thanks for listening!