Prelim Workshop
Summer 2018

Algebra Worksheet 5: Linear Algebra II

Eigenspaces and diagonalization

- Geometric $\leq$ algebraic multiplicity of an eigenvalue. Technique to prove: triangularization ($*$)
- $A \in M_n(F)$ diagonalizable over F if and only if minimal polynomial splits in F with no repeated roots.
- $A \in M_n(F)$ upper-triangularizable over F if and only if characteristic polynomial splits in F.
- Spectral theorems: real symmetric matrices diagonalizable over $\mathbb{R}$ by an orthogonal matrix. Complex Hermitian (more generally, normal) matrices diagonalizable over $\mathbb{C}$ by a unitary matrix.
- Commuting diagonalizable matrices are simultaneously diagonalizable. Technique to prove: invariant subspaces. (7.6.17 then $**$, 7.5.17)

Jordan and Rational Canonical Forms - see “Prelim Workshop Lecture Notes” for review

- Uniqueness, conditions for existence
- Jordan form and diagonalizability
- Computing the Jordan form (7.6.24, 7.6.30)
- Relationship of JCF and RCF to minimal and characterstic polynomials. (What is the minimal polynomial of a block-diagonal matrix?)
- Using JCF and RCF to determine similarity (7.7.6, 7.7.10)
- Using Jordan form to study asymptotics of A^n (7.7.19)
- Jordan form and matrix nth roots (7.6.7)

7.5.17 • Let S be a nonempty commuting set of $n \times n$ complex matrices ($n \geq 1$). Prove that the members of S have a common eigenvector.

7.6.7 •. Prove or disprove: For any 2×2 matrix A over $\mathbb{C}$, there is a 2×2 matrix B such that $A = B^2$.

7.6.17 •. Let V be a finite-dimensional vector space and $T : V \to V$ a diagonalizable linear transformation. Let $W \subseteq V$ be a linear subspace which is mapped into itself by T. Show that the restriction of T to W is diagonalizable.

7.6.24 • Find the eigenvalues, eigenvectors, and the Jordan canonical form of

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

considered as a matrix in $F_3 = \mathbb{Z}/3\mathbb{Z}$.

7.6.30 Find the Jordan canonical form of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

7.7.6 Let A and B be $n \times n$ matrices over a field F such that $A^2 = A$ and $B^2 = B$. Suppose that A and B have the same rank. Prove that A and B are similar.

7.7.10 Let A and B be two real $n \times n$ matrices. Suppose there is a complex invertible $n \times n$ matrix U such that $A = UBU^{-1}$. Show that there is a real invertible $n \times n$ matrix V such that $A = VBV^{-1}$. (In other words, if two real matrices are similar over $\mathbb{C}$, then they are are similar over $\mathbb{R}$.)

7.7.19 Let A be a complex $n \times n$ matrix such that the sequence $(A^n)_{n=1}^{\infty}$ converges to a matrix B. Prove that B is similar to a diagonal matrix with zeros and ones along the main diagonal.

For the following two questions, do not cite the existence of the Jordan form.

($*$) • Prove that the geometric multiplicity of an eigenvalue is less than or equal to its algebraic multiplicity.

($**$) •. Say A and B are diagonalizable and $AB = BA$. Prove that there exists an invertible matrix T such that TAT^{-1} and TBT^{-1} are both diagonal.