Prelim Workshop
Summer 2018

Analysis Worksheet 5: Complex Analysis I

1. Prove that the uniform limit of a sequence of complex analytic functions is complex analytic. Is the analogous theorem true for real analytic functions?

   SdS 5.2.16, Sp78. Morera's Theorem. Stone-Weierstrass Theorem.

2. Prove that there is no one-to-one conformal map of the punctured disk $G = \{z \in \mathbb{C} : 0 < |z| < 1\}$ onto the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

   SdS 5.3.8, Sp95. Removable singularities, Open Mapping Theorem.

3. Let $f(z)$ be a function that is analytic in the unit disk $\mathbb{D} = \{|z| < 1\}$. Suppose that $|f(z)| \leq 1$ in $\mathbb{D}$. Prove that if $f(z)$ has at least two fixed points z_1 and z_2, then $f(z) = z$ for all $z \in \mathbb{D}$.

   SdS 5.4.7, Sp03. Schwarz Lemma.

4. Suppose that f is analytic on the open upper half-plane and satisfies $|f(z)| \leq 1$ for all z, $f(i) = 0$. How large can $|f(2i)|$ be under these conditions?

   SdS 5.4.13, Fa79, Fa90. Schwarz Lemma, common conformal maps.

5. Let the rational function f in the complex plane have no poles for $Im(z) \geq 0$. Prove that
   $$\sup\{|f(z)| : Im(z) \geq 0\} = \sup\{|f(z)| : Im(z) = 0\}.$$

   SdS 5.5.3, Fa99. Maximum Modulus Principle.

6. Let f and g be two entire functions such that, for all $z \in \mathbb{C}$, $Re(f(z)) \leq kRe(g(z))$ for some real constant k (independent of z). Show that there are constants a, b such that
   $$f(z) = ag(z) + b.$$

   SdS 5.5.8, Sp97. Liouville's Theorem.