Prelim Workshop
Summer 2018

Analysis Worksheet 2: Metric Spaces

1. Let $X \subseteq \mathbb{R}^n$ be compact and let $f : X \to \mathbb{R}$ be continuous. Given $\epsilon > 0$, show there is an M such that for all $x, y \in X$,
$$|f(x) - f(y)| \leq M|x - y| + \epsilon.$$
SdS 4.1.18, Fa89. Uniform continuity.

2. Let K be a continuous function on the unit square $0 \leq x, y \leq 1$ satisfying $|K(x, y)| < 1$ for all x and y. Show that there is a continuous function $f(x)$ on $[0, 1]$ such that we have
$$f(x) + \int_0^1 K(x, y)f(y)\, dy = e^{x^2}.$$
Can there be more than one such function f?
SdS 4.3.5, Fa82. Contraction Principle, Fixed Point Theorems.

3. Let X be a compact metric space and $f : X \to X$ an isometry. Show that $f(X) = X$.
SdS 4.2.6, Fa80. Sequential compactness.

4. Let F be a uniformly bounded, equicontinuous family of real valued functions on the metric space (X, d). Prove that the function
$$g(x) = \sup\{f(x) : f \in F\}$$
is continuous.
SdS 4.2.10, Sp87. Equicontinuity, Arzela-Ascoli Theorem.

5. Let $X \subseteq \mathbb{R}^n$ be a closed set and r a fixed positive real number. Let $Y = \{y \in \mathbb{R}^n : |x - y| = r, \text{some } x \in X\}$. Show that Y is closed.
SdS 4.1.13, Fa89. Sequential definitions.