Prelim Workshop
Summer 2018

Analysis Worksheet 1: Differential Equations

1. Suppose f is a differentiable function from the reals into the reals. Suppose $f'(x) > f(x)$ for all $x \in \mathbb{R}$, and $f(x_0) = 0$. Prove that $f(x) > 0$ for all $x > x_0$.

   SdS 1.4.8, Sp77, Su82. Gronwall inequality.

2. Let n be an integer larger than 1. Is there a differentiable function on $[0, \infty)$ whose derivative equals its nth power and whose value at the origin is positive?

   SdS 3.1.3, Fa77, Fa93. ODE ill-posedness.

3. Prove that the initial value problem
$$\frac{dx}{dt} = 3x + 85\cos x, \qquad x(0) = 77,$$
   has a solution $x(t)$ defined for all $t \in \mathbb{R}$.

   SdS 3.1.9, Su77, Su80, Sp82, Sp83. Picard's Theorem (ODE well-posedness).

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous nowhere vanishing function, and consider the differential equation
$$\frac{dy}{dx} = f(y).$$
   1. For each real number c, show that this equation has a unique continuously differentiable solution $y = y(x)$ on a neighborhood of 0 which satisfies the initial condition $y(0) = c$.
   2. Deduce the conditions on f under which the solution y exists for all $x \in \mathbb{R}$, for every initial value c.

   SdS 3.1.10, Fa82. Implicit/Inverse Function Theorem. (Previous incorrect suggestion: Fixed Point Theorem.)

5. Consider the equation
$$\frac{dy}{dx} = y - \sin y.$$
   Show that there is an $\epsilon > 0$ such that if $|y_0| < \epsilon$, then the solution $y = f(x)$ with $f(0) = y_0$ satisfies
$$\lim_{x \to -\infty} f(x) = 0.$$

   SdS 3.1.14, Sp84.