Virtual properties of 3-manifolds

dedicated to the memory of Bill Thurston

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Abstract. We will discuss the proof of Waldhausen's conjecture that compact aspherical 3-manifolds are virtually Haken, as well as Thurston's conjecture that hyperbolic 3-manifolds are virtually fibered. The proofs depend on major developments in 3-manifold topology of the past decades, including Perelman's resolution of the geometrization conjecture, results of Kahn and Markovic on the existence of immersed surfaces in hyperbolic 3-manifolds, and Gabai's sutured manifold theory. In fact, we prove a more general theorem in geometric group theory concerning hyperbolic groups acting on CAT(0) cube complexes, concepts introduced by Gromov. We resolve a conjecture of Dani Wise about these groups, making use of the theory that Wise developed with collaborators including Bergeron, Haglund, Hsu, and Sageev as well as the theory of relatively hyperbolic Dehn filling developed by Groves-Manning and Osin.

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1. Introduction

In Thurston's 1982 Bulletin of the AMS paper *Three Dimensional Manifolds, Kleinian groups, and hyperbolic geometry* [118], he asked 24 questions which have guided the last 30 years of research in the field. Four of the questions have to do with "virtual" properties of 3-manifolds:

- Question 15 (paraphrased): Are Kleinian groups LERF? [76, Problem 3.76 (Hass)]
- Question 16: "Does every aspherical 3-manifold have a finite-sheeted cover which is Haken?" This question originated in a 1968 paper of Waldhausen. [75, Problem 3.2] [1]
- Question 17: "Does every aspherical 3-manifold have a finite-sheeted cover with positive first Betti number?" [76, Problem 3.50 (Mess)]

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[1] *"Of those irreducible manifolds, known to me, which have infinite fundamental group and are not sufficiently large, some (and possibly all) have a finite cover which is sufficiently large."* [122] Waldhausen may only have been referring to small Seifert-fibered space examples that he was aware of, but the general question has been attributed to him.

- Question 18: "Does every hyperbolic 3-manifold have a finite-sheeted cover which fibers over the circle? This dubious-sounding question seems to have a definite chance for a positive answer." [76, Problem 3.51 (Thurston)]

The goal of this talk is to explain these problems, and how they reduce to a conjecture of Wise in geometric group theory.

Note that there are now several expository works on the topics considered here [21, 18, 19, 30, 46].

2. 3-manifold topology

Haken introduced the notion of a Haken manifold as a way to understand certain 3-manifolds via an inductive procedure by cutting along surfaces [66].

Definition 2.1. A *closed essential surface* $f : \Sigma^2 \to M^3$ is a surface with either

- $\chi(\Sigma) \leq 0$ and $f_\# : \pi_1(\Sigma) \hookrightarrow \pi_1(M)$ is injective or
- $\Sigma \cong S^2$, and $[f] \neq 0 \in \pi_2(M)$ (in other words, f is not homotopically trivial).

If M is a manifold, then M is termed *aspherical* if its universal cover $\tilde{M}$ is contractible. For example, this holds if $\tilde{M} \cong \mathbb{R}^n$. In three dimensions, M is closed and aspherical if and only if $\tilde{M} \cong \mathbb{R}^3$, or equivalently $\pi_2(M) = \pi_3(M) = 0$ (this is a non-trivial consequence of the geometrization conjecture). By the sphere theorem of Papakyrokopoulos [103], equivalently $|\pi_1(M)| = \infty$ and M is irreducible.

If M is aspherical and contains an embedded essential surface, then M is called *Haken*.

For example if M is aspherical, and $rank(H_1(M;\mathbb{Q})) = b_1(M) > 0$, then M is Haken. This follows from the loop theorem.

A 3-manifold M **fibers over the circle** if there is a map $\eta : M \to S^1$ such that each point preimage $\eta^{-1}(x)$ is a surface called a **fiber**.

If M is closed and 3-dimensional and fibers over S^1, then the fiber is a genus g surface F_g, and M is obtained as the mapping torus of a homeomorphism $f : F_g \to F_g$ (Figure 1),

$$M \cong T_f = \frac{F_g \times [0,1]}{\{(x,0) \sim (f(x),1)\}}.$$

A fibered 3-manifold M has positive first betti number, and the fiber surface is essential. Therefore M is aspherical if $g > 0$.

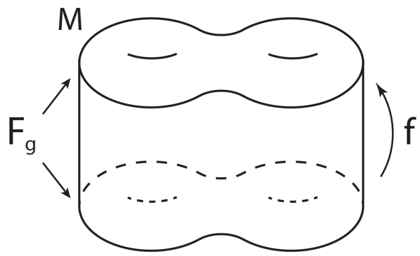
A motivating question in 20th century 3-manifold topology:

Given an immersed essential surface in a 3-manifold, does there exist an embedded essential surface of the same type?

This has been an important question because embedded essential surfaces are easier to work with than immersed surfaces in general. For example, the theory of normal surfaces allows certain questions about embedded essential surfaces in 3-manifolds to be made algorithmic.

Examples include when $\chi(\Sigma) \geq 0$:

Figure 1. A fibered manifold is a mapping torus of a surface homeomorphism $f : F_g \to F_g$



- Dehn's Lemma [41, ~~Dehn 1910~~] [103, Papakyriokopoulos 1957]: If an embedded loop in ∂M is homotopically trivial, then it bounds an embedded disk.
- The Loop Theorem [103]: Similar statement for an immersed loop in ∂M.
- The Sphere Theorem [103, Papakyriokopoulos 1957] [112, Stallings 1969]: If $\pi_2(M) \neq 0$ (i.e., there's an immersed essential sphere in M), then there exists an embedded essential sphere in M.
- The annulus and torus theorems [72, Jaco-Shalen] and [73, Johannson]:

  In a Haken manifold, if there is an immersed essential annulus or torus, then there is an embedded one.
- The Seifert fibered space theorem [Scott [111], Mess, Tukia [121], Casson-Jungreis [35], Gabai [52]]:

  If the center $Z(\pi_1(M)) \neq 0$ and M is aspherical, then M is Seifert-fibered.

As was known to Waldhausen, there is an infinite class of aspherical Seifert-fibered spaces which are non-Haken, so one cannot hope to extend the torus theorem to non-Haken 3-manifolds. For example, one may consider the unit tangent bundle to a turnover orbifold of euler characteristic < 0. However, these are easily shown to be virtually Haken, since they have a finite-sheeted cover homeomorphic to the unit tangent bundle of a surface. Thus, one may ask the question:

Given an immersed essential surface in a 3-manifold, does there exist a finite-sheeted cover with an embedded essential surface of the same type?

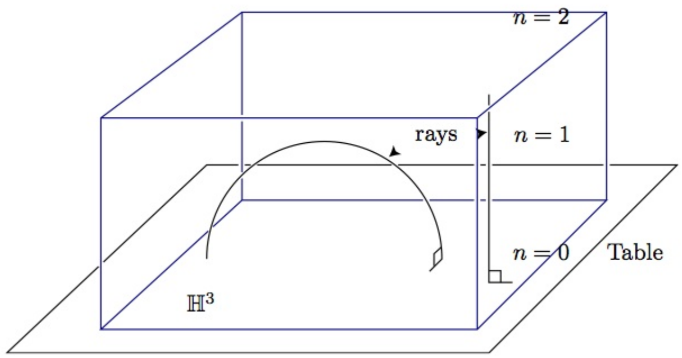
These classic theorems of 3-manifold topology are now superseded by the Geometrization Theorem (Question 1 from Thurston's list [118] [76, Problem 3.45 (Thurston)]). The geometrization theorem states that an irreducible 3-manifold M admits a (possibly non-orientable) embedded essential surface $\Sigma \hookrightarrow M$ which is unique up to isotopy, such that $\chi(\Sigma) = 0$ and each component of $M - \Sigma$ admits a complete locally homogeneous Riemannian metric of finite volume. There are eight possible model geometries for these metrics.

This question was formulated by William Thurston at Princeton in the 1970s, and was proved by him for Haken 3-manifolds [119, 120], and conjectured to hold in general. A proof of the conjecture was given by Grigori Perelman in 2003 using Ricci flow [104], finishing a program of Hamilton who introduced the Ricci flow in the 1980s [68].

The most interesting and least understood homogeneous geometry is hyperbolic geometry.

Consider a chunk of glass sitting on a table, so that the speed of light n is proportional to the height above the table (Figure 2). Then light will follow a geodesic path in the glass which is a semicircle or line perpendicular to the tabletop.

Figure 2. A physical model for hyperbolic space



This gives a physical model for the upper half space model of hyperbolic space.

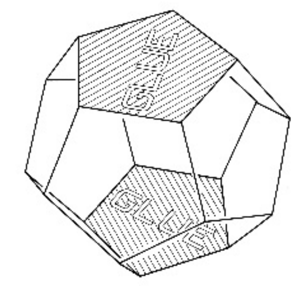
Manifolds modeled on this geometry are *hyperbolic 3-manifolds* if they admit a complete Riemannian metric of constant curvature -1, with fundamental group a *Kleinian group* (if it is finitely generated). Classic examples of hyperbolic 3-manifolds are the **Seifert-Weber dodecahedral space**, the **figure eight knot complement**, and the **Whitehead link complement** (Figure 3).

Given a cusped hyperbolic 3-manifold (finite-volume non-compact), Thurston showed that one may deform the hyperbolic metric to obtain hyperbolic metrics on Dehn fillings [117, Theorem 5.8.2]. A Dehn filling is obtained from a manifold with torus boundary by identifying the boundary with the boundary of a solid torus (Figure 4). The homeomorphism type of the Dehn filling is determined by the slope of the meridian of the torus, which may be regarded as a rational number $\in \mathbb{PQ}^1$.

Thurston proved that all but finitely many slopes $\in \mathbb{PQ}^1$ give Dehn fillings on a hyperbolic 3-manifold are hyperbolic.

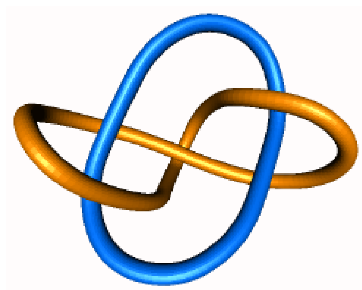
An aspherical 3-manifold M whose geometric decomposition does not contain a hyperbolic piece, then M is called a **graph manifold**. If M is not geometric, then all of the geometric pieces of the JSJ decomposition are modeled on the geometry $\mathbb{H}^2 \times \mathbb{R}$.

Figure 3. Examples of hyperbolic manifolds of finite volume



(a) Seifter-Weber space

(b) Figure 8 knot



(c) Whitehead link

3. Virtual properties of 3-manifolds

- Recall that a compact aspherical 3-manifold M is **Haken** if it contains an embedded π_1-injective surface (e.g. a knot complement). The Seifert-Weber space is non-Haken [28, Burton-Rubinstein-Tillmann], as well as hyperbolic surgeries on the figure 8 knot complement [117, Corollary 4.11].

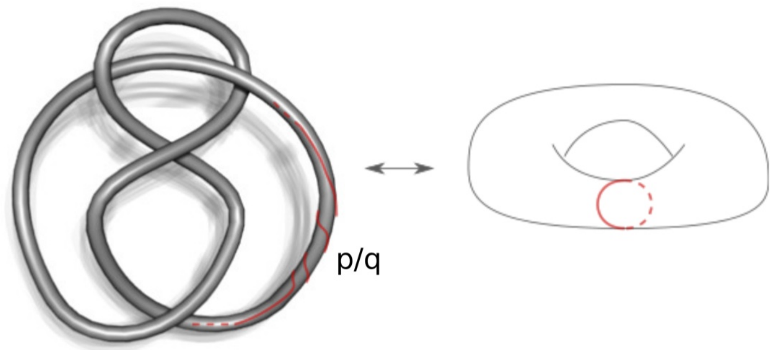
- A 3-manifold M is **virtually Haken** if there is a finite-sheeted manifold cover $\tilde{M} \to M$ such that $\tilde{M}$ is Haken, e.g. hyperbolic surgeries on the figure 8 knot complement are virtually Haken [44, Dunfield-Thurston].

- Waldhausen conjectured that every aspherical 3-manifold M is virtually Haken (the *virtual Haken conjecture*, Question 16).

- A fortiori, does M have a finite-sheeted cover $\tilde{M} \to M$ with $b_1(M) > 0$ (Question 17)? Recall that $b_1(M) = rank(H_1(M; \mathbb{Q}))$.

- There has been much work on the virtual Haken conjecture before for certain classes of manifolds. These include manifolds in the Snappea census [44], surgeries on various classes of cusped hyperbolic manifolds [12, 13, 14, 25,

Figure 4. Dehn filling on the figure 8 knot complement



37, 39, 40, 77, 93, 94], certain arithmetic hyperbolic 3-manifolds (see [109] and references therein), and manifolds satisfying various group-theoretic criteria [78, 79, 87].

Remark: Since closed 3-manifold fundamental groups have balanced presentations, it is unlikely that a generic 3-manifold M has $b_1(M) > 0$, which clarifies the difficulty of this question.

- M is **virtually fibered** if there exists a finite-sheeted cover $\tilde{M} \to M$ such that $\tilde{M}$ fibers.
- If M fibers, then $b_1(M) > 0$, so this is stronger than asking for virtual positive betti number.
- There have previously been several classes of hyperbolic 3-manifolds shown to virtually fiber, including 2-bridge links [123, Walsh], some Montesinos links [4, Agol-Boyer-Zhang],[59, Guo-Zhang], [58, Guo], and certain alternating links [9, Aitchison-Rubinstein], as well as many examples of hyperbolic manifolds [17, Bergeron], [36, Chesebro-DeBlois-Wilton], [49, Gabai], [82, Leininger], [106, Reid], [125, Wise].
- Thurston asked whether every hyperbolic 3-manifold is virtually fibered (Question 18)?

If M is a finite volume hyperbolic 3-manifold, and $f : \S_g \to M$ is an essential immersion of a surface of genus $g > 0$, then there is a dichotomy for the geometric structure of the surface discovered by Thurston, and proven by Bonahon in general [23].

Either f is

- **geometrically finite** or
- **geometrically infinite**.

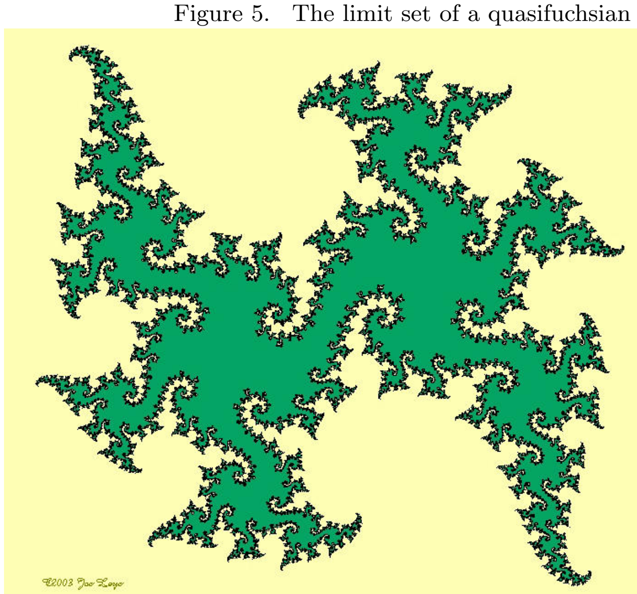
The first case includes **quasifuchsian** surfaces (Figure 5). A geometrically finite surface preserves a convex subset of hyperbolic space whose quotient by the group has finite (non-zero) volume.

In the geometrically infinite case, the surface is **virtually the fiber** of a fibering of a finite-sheeted cover of M.

The **Tameness theorem** [1, Agol], [31, Calegari-Gabai] plus the **covering theorem** of [32, Canary] implies a similar dichotomy for finitely generated subgroups of $\pi_1(M)$:

either a subgroup is geometrically finite, or it corresponds to a virtual fiber.

Figure 5. The limit set of a quasifuchsian surface group

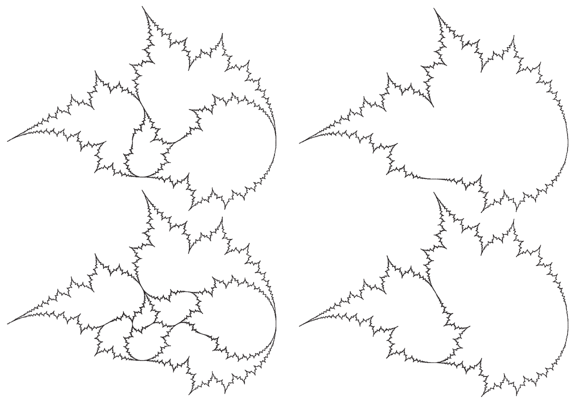


The limit set of a fiber of a fibration is $\partial_\infty \mathbb{H}^3 = \hat{\mathbb{C}}$, but may be regarded as a sphere-filling curve [34, Cannon-Thurston]. In certain cases, one may construct these sphere-filling curves by approximation by subdivision tilings [10, Alperin-Dicks-Porti] (Figure 6) in a similar fashion to the classical construction of Peano curves by approximations.

Analogous to the loop, sphere, annulus and torus theorems, one may ask:

Given an essential map of a surface $f : \Sigma \to M$ with $\chi(\Sigma) < 0$, is there an essential embedding $\Sigma \hookrightarrow M$? The answer to this question is no since there are examples of non-Haken 3-manifolds such as the figure 8 knot hyperbolic fillings which have virtual positive betti number, and therefore contain an immersed essential surface, but no embedded essential surface.

Figure 6. Approximates to the sphere-filling Peano curve invariant under the figure 8 knot complement fiber group



With further hypotheses on the surface, the answer to this question can be a qualified yes.

Gabai proved that if $f : \Sigma \looparrowright M$ is an immersed oriented surface with $\chi(\Sigma) \leq 0$, and $f_*([\Sigma]) \neq 0 \in H_2(M)$, then there is an embedded essential surface $\Sigma' \hookrightarrow M$ such that $[\Sigma'] = f_*[\Sigma] \in H_2(M)$, and $\chi(\Sigma') \geq \chi(\Sigma)$ [48, 50, 51].

Gabai's proof makes use of an inductive method called **sutured manifold hierarchies** to construct a foliation of the manifold with an embedded compact leaf, and obtain the desired lower bound on Euler characteristic by analyzing the Euler class of the foliation.

Theorem 3.1. *[74, Kahn-Markovic] [76, Problem 3.75 (Waldhausen)] Hyperbolic 3-manifolds contain immersed quasi-fuchsian surfaces which are arbitrarily close to being totally geodesic.*

The limit sets of these surfaces in $\partial_\infty \mathbb{H}^3$ can be made arbitrarily close to being a round circle.

There has been much previous work on this problem, including the following results:

- Cooper-Long and Li proved that all but finitely many Dehn fillings on a cusped hyperbolic 3-manifold have essential surfaces [38, Cooper-Long], [84, Li].
- Masters-Zhang showed that cusped hyperbolic 3-manifolds contain essential quasifuchsian surfaces (no parabolics) [95, 96, Masters-Zhang]. Together with [15, Bart 2001], this implies most dehn fillings on multi-cusped manifolds contain essential surfaces.
- Lackenby proved in 2008 that arithmetic hyperbolic 3-manifolds contain closed essential immersed surfaces (using work of Lewis Bowen) [81, 80, Lackenby].

4. 3-manifold fundamental group properties

Definition 4.1. A group G is **residually finite (RF)** if for every $1 \neq g \in G$, there exists a finite group K and a homomorphism $\phi : G \to K$ such that $\phi(g) \neq 1 \in K$.

Alternatively,

$$\{1\} = \bigcap_{[G:H]<\infty} H. \tag{1}$$

Examples of residually finite groups include

- finitely generated linear groups [90, Malcev];
- 3-manifold groups [69, Hempel] + Geometrization [104]; and
- mapping class groups of surfaces [55, Grossman].

Definition 4.2. A subgroup $L < G$ is *separable* if for all $g \in G - L$, there exists $\phi : G \to K$ finite such that $\phi(g) \notin \phi(L)$.

Alternatively,

$$L = \bigcap_{L \leq H \leq G, [G:H]<\infty} H \tag{2}$$

Residual finiteness of G is equivalent to $1 < G$ is separable.

Definition 4.3. A subgroup $L < G$ is *weakly separable* if for all $g \in G - L$, there exists $\phi : G \to K$ such that $\phi(L)$ is finite and $\phi(g) \notin \phi(L)$ (K need not be finite).

Example: f $L < G$ is finite, then L is (trivially) weakly separable in G.

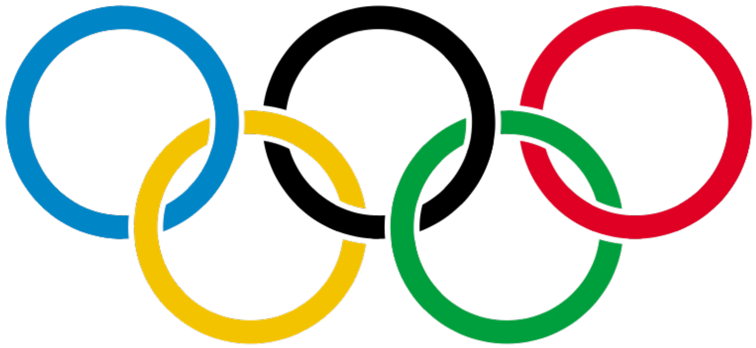
Example: Let $H \lhd G$ be a normal subgroup of G, then H is weakly separable in G. In fact, we may use the quotient $\varphi : G \to G/H$ to weakly separate all elements of $G - H$ from H.

Definition 4.4. A group G is **Locally Extended Residually Finite (LERF)** if *finitely generated subgroups of* G *are separable.* (*local* means finitely generated)

Previously well-known examples of LERF groups include

- $\mathbb{Z}^n$;
- free groups [67, Hall] and surface groups [110, Scott];
- doubles of certain compression body groups [53, Gitik];
- Bianchi groups $\mathrm{PSL}(2, \mathbb{Z}[\sqrt{-d}])$ [7, Agol-Long-Reid] and certain other arithmetic subgroups of $\mathrm{PSL}(2, \mathbb{C})$ such as the fundamental group of the Seifert-Weber dodecahedral space;
- 3-dimensional hyperbolic reflection groups [64, Haglund-Wise].

Figure 7. A link whose complement does not have LERF fundamental group



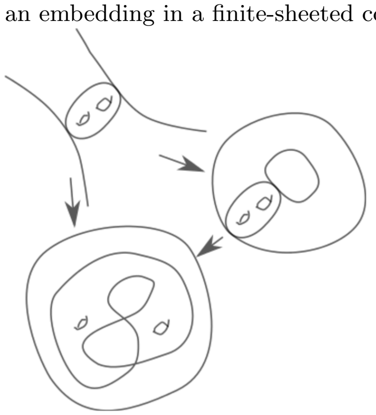
There are examples of 3-manifold groups which are not LERF which are *graph manifold* groups [27, Burns-Karrass-Solitar].

Thurston's question 15 is whether Kleinian groups are LERF?

For example, the fundamental group of the complement of the link in Figure 7 is not LERF [100, Niblo-Wise].

LERF allows one to lift π_1-injective immersions to embeddings in finite-sheeted covers [110, Scott].

Figure 8. A surface immersed in a 3-manifold with separable fundamental group lifts to an embedding in a finite-sheeted cover



In fact, Matsumoto showed that there are certain graph manifolds which contain surfaces which do not lift to an embedding in any finite-sheeted covering space [97, Matsumoto]. These examples highlight the importance of hyperbolicity with respect to subgroup separability.

4.1. Virtual fibering. Thurston's virtual fibering question was stated for hyperbolic 3-manifolds, and does not hold for general 3-manifolds.

Theorem 4.5 (Przytycki-Wise 2012). *If M is an aspherical closed 3-manifold which is not a graph manifold, then M is virtually fibered.*

Svetlov characterized virtually fibered graph manifolds (e.g. unit tangent bundles to closed hyperbolic surfaces are not virtually fibered), but the criterion is technical to state [116, Svetlov].

Definition 4.6. A group G is **Residually Finite Rationally Solvable** or **RFRS** if there is a sequence of subgroups $G = G_0 > G_1 > G_2 > \cdots$ such that $\cap_i G_i = \{1\}$, $[G : G_i] < \infty$ and $G_{i+1} = ker\{G_i \to \mathbb{Z}^{k_i} \to (\mathbb{Z}/n_i)^{k_i}\}$ for sequences $n_i, k_i \in \mathbb{N}$.

Remark: We may assume that $G_i \lhd G$, in which case G/G_i is a finite solvable group. Thus, the RFRS condition is a strong form of residual finite solvability.

We remark that if G is RFRS, then any subgroup $H < G$ is as well.

Examples of RFRS groups are free groups, surface groups, $\mathbb{Z}^n$ and free products of RFRS groups.

For a 3-manifold M with RFRS fundamental group, the condition is equivalent to there existing a **cofinal tower** of finite-index covers

$$M \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots$$

such that M_{i+1} is obtained from M_i by taking a finite-sheeted cyclic cover dual to an embedded non-separating surface in M_i. Equivalently, $\pi_1(M_{i+1}) = ker\{\pi_1(M_i) \to \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}\}$.

This condition implies that M has virtual infinite b_1, unless $\pi_1(M)$ is virtually abelian.

Theorem 4.7. *[2, Agol] If M is aspherical and $\pi_1(M)$ is RFRS, then M virtually fibers.*

The proof makes use of **sutured manifold theory**, the inductive technique mentioned before for studying foliations of 3-manifolds introduced by Gabai. For a self-contained proof, see a preprint of [46, Friedl-Kitayama].

Theorem 4.8. *[125, Corollary 14.3, Theorem 14.29] Haken hyperbolic 3-manifolds are virtually fibered.*

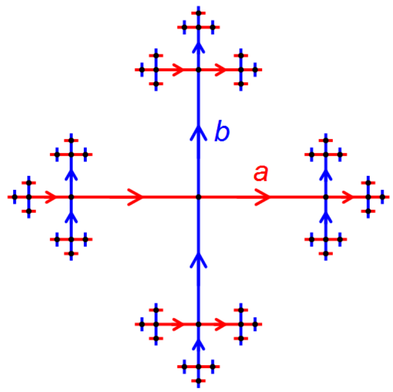
The theorem includes non-compact hyperbolic 3-manifolds with finite volume unconditionally.

5. Geometric group theory

Let G be a finitely generated group, with generators $G = \langle g_1, \ldots, g_n \rangle$. The *Cayley graph* of G with respect to the generating set $\{g_1, \ldots, g_n\}$ is a graph $\Gamma = \Gamma(G, \{g_1, \ldots, g_n\})$ with vertex set $V(\Gamma) = G$, and edge set $E(\Gamma) = \{(g, g \cdot g_i) | g \in G, 1 \leq i \leq n\}$. So the degree of each vertex g is $2n$.

We may regard Γ as a metric space, by letting edges of Γ have length 1, and taking the path metric. So the distance $d(1, g)$ between vertices $1, g \in V(\Gamma)$ is the smallest k such that $g = g_{i_1}^{\pm 1} \cdots g_{i_k}^{\pm 1}$. Then clearly $d(h, h \cdot g) = d(1, g)$, since the metric is invariant under the left group action of G on $\Gamma(G, \{g_1, \ldots, g_n\})$.

Here is a picture of the Cayley graph $\Gamma(F_2, \{a, b\})$ of the two generator free group $F_2 = \langle a, b \rangle$ with respect to the free generating set $\{a, b\}$:



Geometric group theory is the study of properties of groups from the geometric properties of the Cayley graph. This notion has some origins in the work of [42, Dehn 1911] on the word problem for surface groups, but was introduced by [99, Milnor 1968] who studied the growth of balls in Cayley graphs of groups as a function of the radius, and [33, Cannon 1984] who studied the Cayley graphs of hyperbolic manifolds.

If G acts properly and cocompactly on a metric space X (for example, $X = \tilde{M}$ the universal cover, where M is a compact Riemannian manifold, and $G = \pi_1(M)$), then some geometric properties of X are reflected in the geometric properties of the Cayley graph $\Gamma(G, \{g_1, \ldots, g_n\})$. So we may study properties of a group G by studying the geometric properties of X.

For example, Milnor observed that if the volumes of balls of radius r in X grow exponentially with r, then the same will hold for the balls in Γ, with volume replaced by the number of vertices. Exponential growth of balls holds for universal covers of compact Riemannian manifolds with negative curvature.

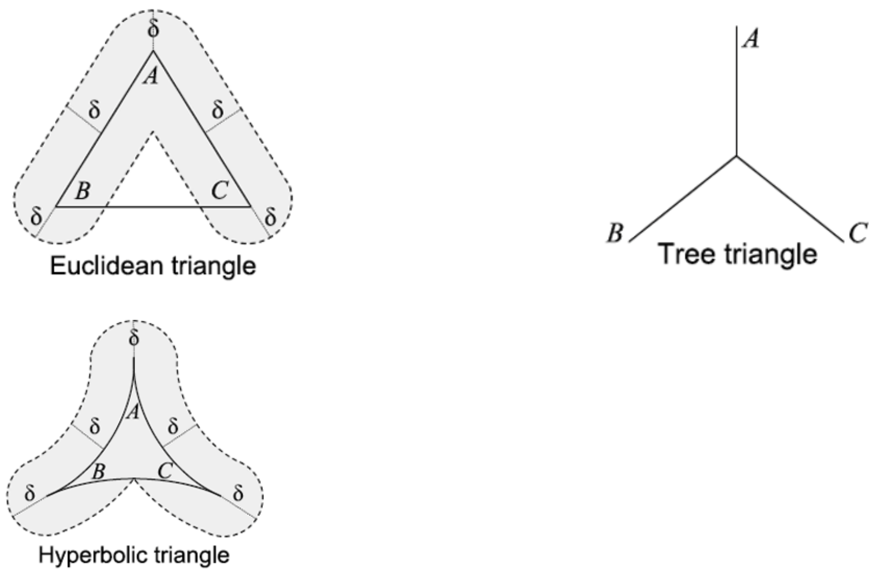
Cannon '84 realized that Cayley graphs of hyperbolic manifolds have a nice recursive combinatorial structure for the balls of radius r. This notion was then extended and codified by [54, Gromov 1987] in the notion of a *hyperbolic group*.

A (Gromov-)hyperbolic geodesic metric space X may be defined by Rips' "slim triangle" condition: for points A, B in the metric space, let $[A, B] \subset X$ be a geodesic connecting A and B. Then X is called a δ-hyperbolic metric space if for any three points $A, B, C \in X$,

$$[B, C] \subset \mathcal{N}_\delta([A, B] \cup [A, C]).$$

For example, hyperbolic space $\mathbb{H}^n$ is $\log(1 + \sqrt{2})$-hyperbolic and a tree is 0-hyperbolic.

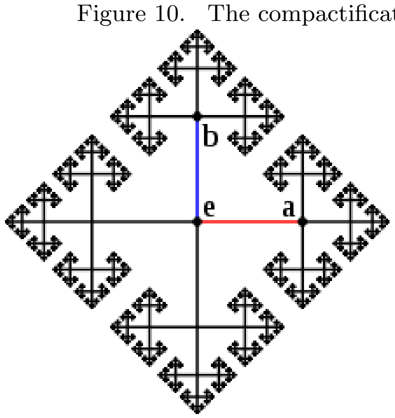
Figure 9. Euclidean vs. slim triangles



If $\Gamma(G, \{g_1, \ldots, g_n\})$ is a δ-hyperbolic metric space for some δ, then G is called a *(Gromov)-hyperbolic* group (sometimes also called δ-hyperbolic, word-hyperbolic, or just hyperbolic group).

Gromov proved many properties of these groups, such as there exists a compactification $\Gamma(G, \{g_1, \ldots, g_n\}) \cup \partial_\infty(G)$, so that $\partial_\infty(G)$ is independent of the generating set and Γ (Figure 10).

Figure 10. The compactification of the Cayley graph of a free group $\langle a, b \rangle$



Definition 5.1. Let X be a geodesic metric space, and $Y \subset X$. Then Y is R-quasiconvex in X if for every $y_1, y_2 \in Y$, the geodesic $[y_1, y_2] \subset X$ lies in an

R-neighborhood of Y, $[y_1, y_2] \subset \mathcal{N}_R(Y)$.

For example, X is δ-hyperbolic if $[a,b]\cup[b,c]$ is δ-quasiconvex for every $a,b,c \in X$.

Let G be a hyperbolic group, with Cayley graph Γ. A subgroup $H < G$ may be regarded as a subspace $H \subset G = V(\Gamma) \subset \Gamma$. Then we say H is *quasiconvex* if it is R-quasiconvex in Γ for some R. It follows from quasigeodesic stability that H will be quasiconvex in the Cayley graph with respect to any (finite) generating set of G.

Motivating examples of hyperbolic groups are Kleinian groups without $\mathbb{Z}^2$ subgroups (e.g. fundamental groups of closed hyperbolic manifolds and convex cocompact Kleinian groups), and more generally fundamental groups of closed negatively curved manifolds. Motivating examples of quasi-convex subgroups are quasi-fuchsian surface groups (such as the fundamental groups of the essential Kahn-Markovic surfaces) in closed hyperbolic 3-manifold groups, and cyclic subgroups of arbitrary hyperbolic groups.

Theorem 5.2. *[5, 91, Agol, Groves, Manning, Martinez-Pedrosa 2008] If* **hyperbolic groups** *are RF, then Kleinian groups are LERF*

So it may be possible to show that hyperbolic 3-manifold groups are LERF by showing that Gromov-hyperbolic groups are RF

Caveat: This approach seems quite unlikely to work, since many experts believe that there are non-RF Gromov-hyperbolic groups.

6. Cube complexes

A topological space X is **locally CAT(0) cubed** if X is a cube complex such that putting the standard Euclidean metric on each cube gives a locally CAT(0) metric (a form of non-positive curvature). Gromov [54] showed that this metric condition is equivalent to a purely combinatorial condition on the links of vertices of X, they are **flag**. A flag simplicial complex has the property that its simplices are determined by the 1-skeleton: if one sees a $k+1$ complete subgraph in the 1-skeleton, then there is a k-simplex spanning the subgraph. If X is locally CAT(0) and simply-connected, then it is globally CAT(0).

In a locally CAT(0) cube complex, there are canonical maps of codimension-one locally geodesic subcomplexes $W \looparrowright X$ called **hyperplanes**, which are obtained by taking the union of midplanes in each cube (Figure 11). The components of the hyperplane complex correspond to equivalence classes of an equivalence relation on edges of the complex generated by edges lying on opposite sides of a square.

A locally CAT(0) square complex has the property that the link of each vertex is a graph of girth ≥ 4 (there are no triangles). In this picture of a square complex, the link of each vertex is a 5-cycle, so it is a CAT(0) square complex (Figure 12).

A topological space Y is **cubulated** if it is *homotopy equivalent* to a compact locally CAT(0) cube complex $X \simeq Y$ (equivalently, Y is aspherical and $\pi_1(X) \cong$

Figure 11. A hyperplane is obtained by extending midplanes

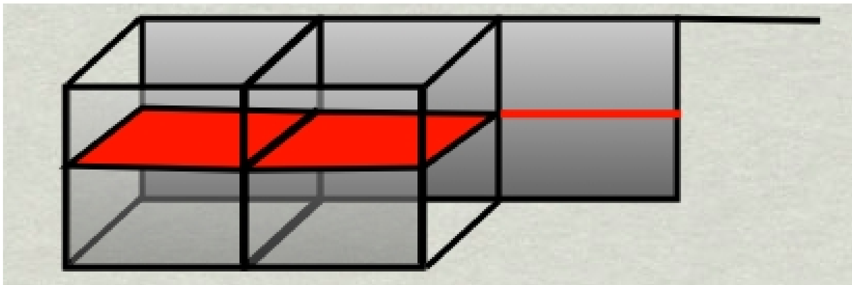
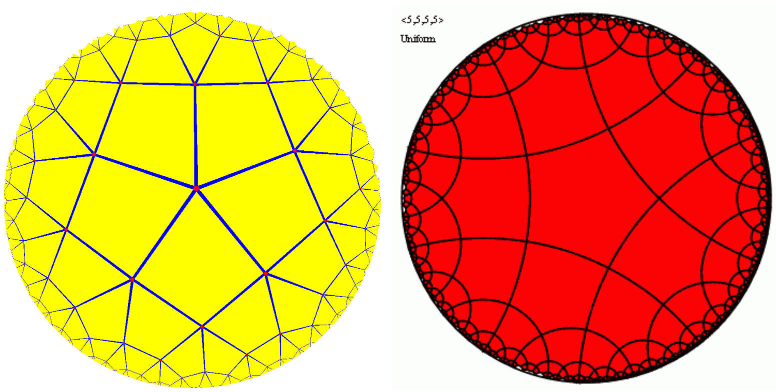


Figure 12. A 2-dimensional CAT(0) cube complex and its hyperplanes



$\pi_1(Y)$). We also say in this case that $\pi_1(Y)$ is cubulated. We are interested in 3-manifolds which are cubulated.

Remark: If $Y = M^3$, and $X \simeq Y$ is a CAT(0) cubing, then $dimX$ may be > 3. Tao Li has shown that there are hyperbolic 3-manifolds Y such that there is no **homeomorphic** CAT(0) cubing $X \cong Y$ [83].

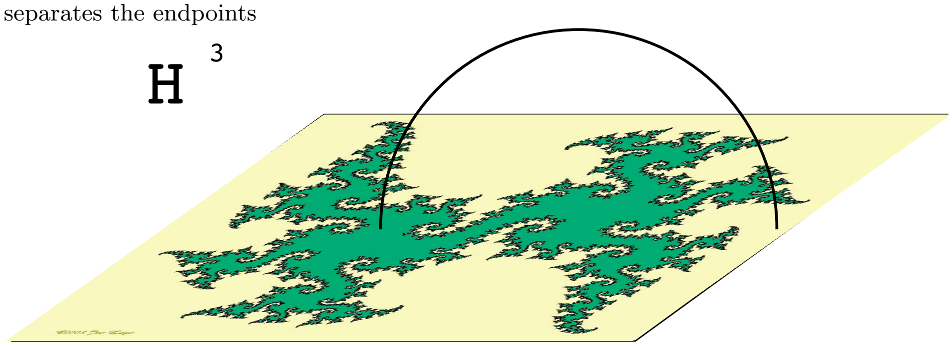
A theorem of [108, Sageev 1995] associates a cocompact action of $\pi_1(M)$ on a (globally) CAT(0) cube complex if M contains an immersed essential surface. Sageev's construction gives a cube complex in which each immersed essential surface in a 3-manifold corresponds to an immersed hyperplane.

For example, Sageev's construction applied to a fiber surface gives an action factoring through the $\mathbb{Z}$ action on $\mathbb{R}$, with quotient S^1. In the case of a geometrically infinite surface in a hyperbolic 3-manifold, Sageev's construction gives rise to a crystallographic group action.

Theorem 6.1. *[20, Bergeron-Wise 2012] Closed hyperbolic 3-manifolds are cubulated.*

Bergeron-Wise give a condition for cubulation. If every geodesic in $\mathbb{H}^3$ has the property that its endpoints in $\partial_\infty \mathbb{H}^3$ are separated by the limit set of a quasifuch-

Figure 13. For the endpoints of each geodesic, there is a quasifuchsian limit set which separates the endpoints

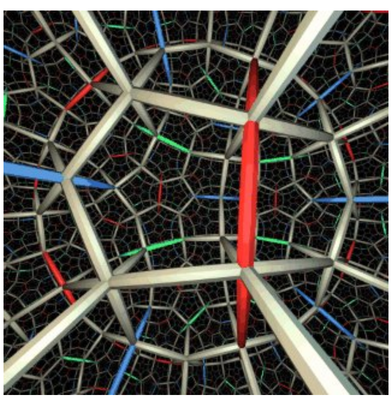


sian surface, then one may use finitely many surfaces so that Sageev's construction will give a proper cocompact action on a CAT(0) cube complex (Figure 13).

The surfaces produced by Kahn-Markovic have limit sets which are close to any given circle, so can separate any pair of points in $\partial_\infty \mathbb{H}^3$. Thus closed hyperbolic 3-manifolds are cubulated.

There were many known examples of cubulated hyperbolic 3-manifolds before this theorem, e.g. alternating link complements [8, Aitchison-Rubinstein]. Other examples come from tessellations by right-angled polyhedra:

Figure 14. The dual tessellation to a cube tessellation of $\mathbb{H}^3$



6.1. Right angled Artin groups.

Definition 6.2. Let Γ be a simplicial graph. The **right-angled Artin group** A_Γ **(RAAG)** defined by Γ has a generator for each vertex $v \in V(\Gamma)$, and relators $vw = wv$ if $(v, w) \in E(\Gamma)$ is an edge of Γ.

The **Salvetti complex** S_Γ associated to A_Γ is a $K(A_\Gamma, 1)$ which is a locally

Figure 15. Some graphs with their associated RAAGs

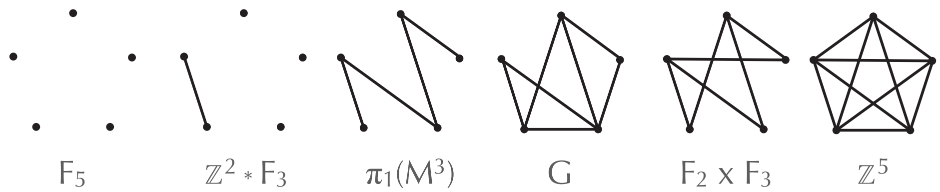
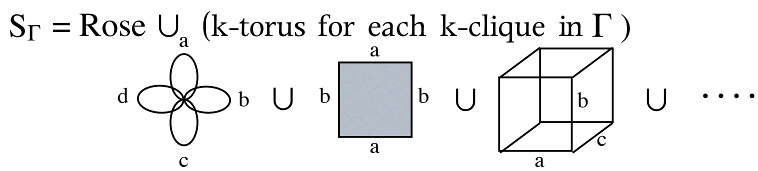


Figure 16. Defining the Salvetti complex S_Γ



CAT(0) cube complex, defined by taking a wedge of loops (rose), one for each generator, and attaching a k-torus for each complete subgraph (k-clique) of Γ (Figure 16). The 2-skeleton by construction gives a presentation $\pi_1(S_\Gamma) \cong A_\Gamma$.

The Salvetti complex has the property that the links of the vertices are flag simplicial complexes, and therefore these complexes are locally CAT(0).

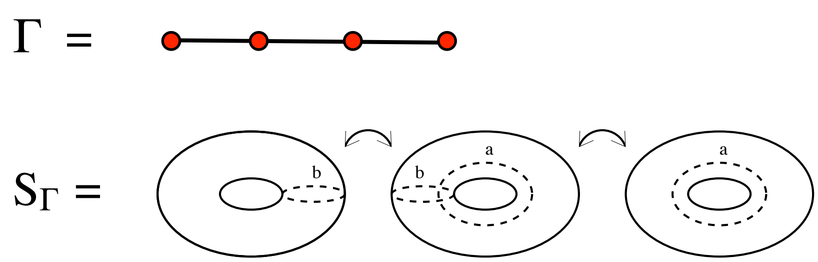
Examples include

- The free group associated to the trivial graph Γ with no edges, for which S_Γ is a wedge of loops
- The n-torus associated to the complete graph on n vertices K_n, for which $S_{K_n} \cong T^n$ the n-torus
- The complement of a chain of 4 links (Figure 17).

6.2. Special cube complexes.

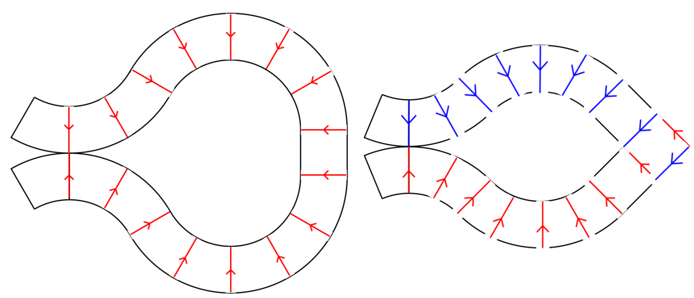
Special cube complexes are defined in terms of properties of their hyperplanes. Hyperplanes are embedded and 2-sided. Moreover, there are no self-osculating or inter-osculating hyperplanes (Figure 18). The midplane of a cube is dual to the edges of the cube it crosses. Thus, we may regard a hyperplane as an equivalence class of (oriented) edges generated by the equivalence relation of two edges lying on opposite sides of a square. If the orientation of edges dual to a hyperplane is preserved in an equivalence class, then the hyperplane is said to be 2-sided. If no adjacent edges of a square are in the same equivalence class, then the hyperplane is embedded. If equivalent edges share a common vertex, which is the end or beginning of both edges, then we say that the hyperplane osculates (Figure 18 (a)). If two hyperplanes osculate at one vertex,

Figure 17. The complement of a chain of 4 links



and cross at another vertex, then we say that the hyperplanes interosculate (Figure 18 (b)). These are the forbidden configurations in a special cube complex.

Figure 18. Configurations forbidden in a special cube complex



(a) Self-osculation (b) Inter-osculation

The motivating examples of special cube complexes are Salvetti complexes of RAAGs.

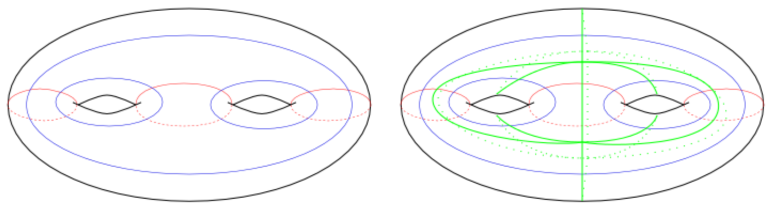
Here's an example of a special cube complex X homeomorphic to a surface. The hyperplanes consist of six curves colored blue and red in Figure 19.

The *crossing graph* $\Gamma(X)$ of a cube complex X has vertices corresponding to the hyperplanes of X, and two vertices of $\Gamma(X)$ are connected by an edge if and only if the corresponding hyperplanes of X cross (Figure 20).

Theorem 6.3. *[62, Haglund-Wise 2007] If X is a special cube complex with hyperbolic fundamental group $\pi_1 X$ (in the sense of Gromov), then $\pi_1 X$ embeds in a RAAG $A_{\Gamma(X)}$ and quasi-convex subgroups of $\pi_1 X$ are separable.*

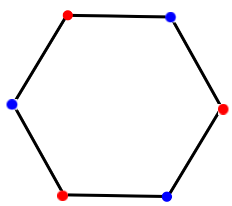
For the proof, take the crossing graph $\Gamma(X)$ associated to X and form the RAAG $A_{\Gamma(X)}$. There is a natural map from X to the Salvetti complex $S_{\Gamma(X)}$ sending every edge dual to a hyperplane to the corresponding edge in the Salvetti

Figure 19. A special cube complex X homeomorphic to a surface



(a) The hyperplanes (b) The green square complex

Figure 20. The crossing graph of X



complex, and extending over the higher skeleta. This map is a locally isometric immersion when X is special, and therefore $\pi_1(X) \leq A_{\Gamma(X)}$.

For example, applying this construction to S_Γ just recovers the identity isometry $S_\Gamma \to S_\Gamma$ and the isomorphism $\pi_1(S_\Gamma) \cong A_\Gamma$!

The notion of a virtual retract was defined independently by [89, Long-Reid] and [61, Haglund]:

Definition 6.4. A subgroup $L < G$ is a virtual retract if there exists $G' < G$ a finite-index subgroup such that $L < G'$ and a retract $r : G' \to L$, meaning $r_{|L} = Id$.

E.g. if $\Lambda \subset \Gamma$ is a subgraph spanned by vertices, there is a natural retract $A_\Gamma \to A_\Lambda$, setting generators of A_Γ corresponding to vertices not in Λ to 1.

Claim: If G is residually finite, and L is a virtual retract of G, then L is separable in G.

Haglund proved that quasi-convex subgroups of RAAGs are virtual retracts [61]. This is proved geometrically using "canonical completions" and "canonical retracts".

Theorem 6.5. *[2, Agol 2008] If M^3 is special cubulated, then M is virtually fibered.*

Since M is special cubulated, $M \simeq X$, where X is a CAT(0) compact special cube complex. Thus $\pi_1(M) = \pi_1(X) < A_{\Gamma(X)}$, a Right-Angled Artin Group. The

RAAGs have the RFRS property, so it passes to $\pi_1(M)$ and implies that M is virtually fibered.

We resolved a conjecture of Wise which implies Thurston's questions.

Theorem 6.6. *[125, Conjecture 19.5 Wise] [3, Agol 2012] Locally CAT(0) cube complexes with hyperbolic fundamental group are virtually special.*

The importance of hyperbolicity in the hypotheses of this theorem is made apparent by the following remarkable theorem:

Theorem 6.7. *[26, Burger-Mozes 1997] There are simple groups which are the fundamental group of a locally CAT(0) square complex whose universal cover is a product of finite-valence regular trees.*

Corollary 6.8. *Let M be a closed hyperbolic 3-manifold. Then $\pi_1 M$ is LERF, large, and M virtually fibers.*

A group G is *large* if there is a finite-index subgroup $G' < G$ which surjects a free group on 2 generators.

This resolves positively Thurston's questions 15-18. The next sections will discuss the background needed in the proof of Theorem 6.6. We remark that the proof of Theorem 6.6 makes use of ideas introduced in the context of 3-manifold topology, including hierarchies and relatively hyperbolic Dehn filling. However, to prove the theorem, it is *essential* that one work in the category of hyperbolic groups, rather than specialize to hyperbolic 3-manifold groups which are of interest for Thurston's questions.

6.3. Amalgamated products and HNN extensions.

Given groups A, B, C, and injections $\varphi_1 : C \hookrightarrow A, \varphi_2 : C \hookrightarrow B$, we may form the amalgamated product $G = A *_C B$, which has a (relative) presentation $\langle A, B | \varphi_1(c) = \varphi_2(c), c \in C\rangle$. By combinatorial group theory, $A, B, C \hookrightarrow G$ inject.

Similarly, suppose we have two subgroups $B, C < A$, such that there is an isomorphism $\varphi : B \to C$. Then the HNN extension $G = A*_\varphi$ has the presentation $\langle A, t | tct^{-1} = \varphi(c), c \in B\rangle$.

For example, all RAAGs are HNN extensions (more generally, any group G with a surjection $G \to \mathbb{Z}$). For any vertex v of a graph Γ defining a RAAG, one has an associated HNN decomposition, where A is the RAAG $A_{\Gamma - v}$, where $\Gamma - v$ is the subgraph obtained by deleting all edges adjacent to v. The subgroup defined by $link(v)$ is both B and C in this case, where $\varphi = Id$, since the generator corresponding to v in A_Γ commutes with all the elements in $A_{link(v)}$. This HNN decomposition may be realized geometrically by splitting the Salvetti complex S_Γ along the hyperplane dual to the generator corresponding to v.

For example, applying this to the complete graph RAAG A_{K_n}, one obtains the HNN extension $\mathbb{Z}^n = \mathbb{Z}^{n-1}*_{Id}$. Another example is a 3-manifold fibered over S^1, $A = \pi_1(F_g)$, $B = C = A$, and $\varphi : B \to C$ is an isomorphism induced by a homeomorphism $f : F_g \to F_g$. Then $\pi_1(T_f) = A*_\varphi$.

6.4. Quasiconvex hierarchies. The notion of a hierarchy originated in the study of 1-relator groups (the Magnus hierarchy), and in the study of Haken 3-manifolds (a Haken hierarchy).

Definition 6.9. The class of groups $\mathcal{QVH}$ (standing for "Quasiconvex Virtual Hierarchy") are defined inductively by

1. $1 \in \mathcal{QVH}$
2. If $G = A *_C B$ or $G = A*_\varphi$, with $A, B, C \in \mathcal{QVH}$ and quasiconvex in G, then $G \in \mathcal{QVH}$.
3. Let $H < G$ with $[G : H] < \infty$. If $H \in \mathcal{QVH}$ then $G \in \mathcal{QVH}$ (in particular with (1), any finite group $K \in \mathcal{VH}$).

The class of groups $\mathcal{MQH}$ is defined similarly, but we require that C is malnormal in G in (2) as well.

It is not hard to show that if M is a hyperbolic 3-manifold, then $\pi_1(M) \in \mathcal{QVH}$ if and only if M is virtually Haken with a finite-sheeted cover containing an embedded quasifuchsian surface.

Special cube complexes with hyperbolic fundamental group are also in $\mathcal{QVH}$, with hierarchy induced by cutting along hyperplanes.

If we have a closed hyperbolic 3-manifold M fibering over S^1 with fiber Σ, then $\pi_1(\Sigma)$ is not quasi convex in $\pi_1(M)$, so $\pi_1(M)$ is not necessarily contained in $\mathcal{QVH}$.

Theorem 6.10. *[125, Wise 2011] Let $G \in \mathcal{QVH}$. Then G is virtually special. That is, there is a CAT(0) cube complex X so that G acts properly cocompactly on X, and a finite-index subgroup $G' < G$ such that X/G' is a special cube complex.*

Wise showed that one-relator groups with torsion are in $\mathcal{QVH}$. This resolved a conjecture of [16, Baumslag 1967].

6.5. Relatively hyperbolic Dehn filling. Recall that the figure eight knot complement has a complete hyperbolic metric of finite volume. However, the figure eight knot group G is not a hyperbolic group, since it contains the peripheral subgroup $\mathbb{Z}^2 = P < G$ coming from the π_1-injective torus that is the boundary of a tubular neighborhood of the knot.

However, I mentioned that Thurston proved that all but finitely many Dehn fillings on the figure 8 knot complement are closed hyperbolic 3-manifolds. In fact, the core of the solid torus of the Dehn filling is a closed geodesic in the hyperbolic structure on the Dehn filling.

Let $G_{p/q}$ be the fundamental group of p/q Dehn filling on the figure eight knot complement. Then $P \cap ker\{G \to G_{p/q}\} = \langle (p, q)\rangle$. In fact, $ker\{G \to G_{p/q}\}$ is freely generated by conjugates of the subgroup $\langle (p, q)\rangle$.

The group G is not hyperbolic, but it is relatively hyperbolic. Roughly, this means that if we take the coset graph of the subgroup P, then this graph is δ-hyperbolic. This notion was suggested by Gromov, and developed by Bowditch

[24] and Farb [45]. There's an extra condition needed called "bounded coset penetration".

Alternatively, Groves and Manning showed that if one attaches "combinatorial horoballs" to the cosets of the peripheral group P, then the resulting space is δ-hyperbolic if and only if G is relatively hyperbolic to P [56].

For example, if F is a free group, and $h \in F$ is a primitive element, then F is hyperbolic relative to $\langle h \rangle$.

For a relatively hyperbolic group, such as the figure eight knot complement, there is an analogue of Thurston's hyperbolic Dehn filling theorem.

Theorem 6.11. *[56, Groves-Manning] [102, Osin] Let G be a group which is hyperbolic relative to the subgroup P. Then there is a finite set of elements $S \subset P - \{1\}$ so that if $P' \lhd P$ is finite-index with $S \cap P' = \emptyset$, then the quotient $G/ \ll P' \gg$ is a hyperbolic group. Moreover, $P \cap \ll P' \gg = P'$.*

For example, if G is a hyperbolic group, and $h \in G$ is a primitive element, then G is hyperbolic relative to $\langle h \rangle$. Then for all sufficiently large n, $G/ \ll h^n \gg$ will also be a hyperbolic group. This result is due to Gromov (or rather small-cancellation theory), but the relatively hyperbolic Dehn filling theorem vastly generalizes this result.

6.6. MSQT.

We state a special case of the Malnormal Special Quotient Theorem (MSQT):

Theorem 6.12. *[125, Wise 2011] Let G be a virtually special hyperbolic group, and let $h \in G$. Then there exists N such that $G/\langle\langle h^n \rangle\rangle$ is virtually special hyperbolic for all $N|n$.*

Remark: The hyperbolicity of $G/ \ll h^n \gg$ for n large may be proved using relatively hyperbolic Dehn filling.

The general statement of the malnormal special quotient theorem is a bit more technical to state. First we need a definition. A collection of subgroups $\{H_1, \ldots, H_m\} < G$ form an almost malnormal collection provided that for any element $g \in G$ with $|H_i \cap g H_j g^{-1}| = \infty$, we must have $i = j$ and $g \in H_i$. We state a strengthened version of the MSQT:

Theorem 6.13. *[125, Theorem 12.3, Malnormal Special Quotient Theorem (MSQT)], [6] Let G be hyperbolic, virtually special, and $\mathcal{H} = \{H_1, \ldots, H_M\} < G$ a almost malnormal collection of quasi convex subgroups. Then there exists $\dot{H}_i \lhd H_i$ such that for any $H'_i < \dot{H}_i$, such $H'_i \lhd H_i$ and H_i/H'_i is virtually special hyperbolic, the quotient group $\overline{G} = G/ << H'_1, \ldots, H'_m >>$ is virtually special hyperbolic.*

Remarks on the proof: The original version of Wise assumes that $[H_i : H'_i] < \infty$. The hypothesis implies that $(G, \mathcal{H})$ is relatively hyperbolic. Using hyperbolic Dehn filling results of Groves-Manning and Osin, one may conclude that $\overline{G}$ is hyperbolic whenever $H_i/\dot{H}_i$ avoids a finite set of elements by Theorem 6.11. The difficult thing is showing that the quotient is cubulated and virtually special.

What Wise actually proves is that there is a finite-index normal subgroup $G' \lhd G$ which has an induced peripheral structure $(G', \mathcal{H}')$, so that $\mathcal{H}'$ contains representatives of each G' conjugacy class of $H_i \cap G'$. Moreover, he shows that a hyperbolic Dehn filling on $(G', \mathcal{H}')$ admits a malnormal quasiconvex hierarchy, so is in $\mathcal{MQH}$. Then he applies his joint work with Haglund [65] and Hsu [70] to conclude that groups with a malnormal quasiconvex hierarchy are virtually special. One may then choose the Dehn filling of G' to be induced from a Dehn filling of G, and thus the Dehn filling of G will be virtually special. The main difference in the new proof of this theorem in [6] is that we first form a malnormal hierarchy of G' which terminates in copies of $\mathcal{H}'$. This gives a malnormal hierarchy for any Dehn filling of G', giving the same conclusion.

The MSQT is the key result that Wise uses to prove that groups in $\mathcal{QVH}$ are virtually special (Theorem 6.10).

6.7. Weak separability of subgroups. The starting point for applying Wise's results to prove his conjecture is the following result proved in the appendix to the paper:

Theorem 6.14. *[3, Agol-Groves-Manning, Appendix] Let G be a hyperbolic group, and $H < G$ a quasi-convex virtually special subgroup. Then H is weakly separable in G.*

The proof of this result is an inductive argument using relatively hyperbolic Dehn filling. It is a direct generalization of the previously mentioned result (Theorem 5.2) that if hyperbolic groups are residually finite, then quasiconvex subgroups are separable. The proof is by induction on **height** of quasiconvex subgroups, which measures how many conjugates of a subgroup intersect in an infinite group. So finite groups have height zero, almost malnormal groups have height 1. One uses relative hyperbolic Dehn filling to reduce the height, and eventually find a quotient in which the image of the subgroup is finite. Note that the same induction on height is used by Wise in the proof of Theorem 6.10 in order to reduce the case of $\mathcal{QVH}$ to the $\mathcal{MQH}$ case.

Hyperbolicity is used in a crucial way in the proof of this theorem, making it inapplicable for example to the examples of Burger-Mozes (Theorem 6.7).

7. Outline of the proof of Wise's conjecture

The proof of Wise's conjecture (Theorem 6.6) is by induction on dimension. Let X be a compact locally CAT(0) cube complex with $G = \pi_1 X$ hyperbolic. Let $W \looparrowright X$ be the immersed hyperplane complex. Then the maximal dimension of cubes in W is one less than those in X, so by induction we may assume that W is virtually special. Then we may apply weak separability to conclude under these hypotheses that

Theorem 7.1. *There exists $G'' \lhd G$, $G/G'' \cong \mathcal{G}$, $\tilde{X}/G'' \cong \mathcal{X}$ such that $\mathcal{X}$ has 2-sided embedded acylindrical compact hyperplanes.*

The acylindrical hypothesis is equivalent to the condition that the fundamental groups of the hyperplanes of $\mathcal{X}$ form a malnormal collection. If $\mathcal{X} \to X$ were a finite-sheeted cover, then we would be done, since we would have proved that $\pi_1(X)$ is in $\mathcal{QVH}$. However, the proof of the theorem produces an infinite-sheeted regular cover.

Definition 7.2 (Crossing Graph). Let $\Gamma(\mathcal{X})$ be a graph with vertex set $V(\Gamma(\mathcal{X})) = \mathcal{W}$ the hyperplanes of $\mathcal{X}$, and edges $(W_1, W_2) \in E(\Gamma(\mathcal{X}))$ if $W_1 \cap W_2 \neq \emptyset$ or if there is an essential cylinder going between W_1 and W_2.

Definition 7.3 (Coloring space). Let $[n] = \{1, \ldots, n\}$. Let

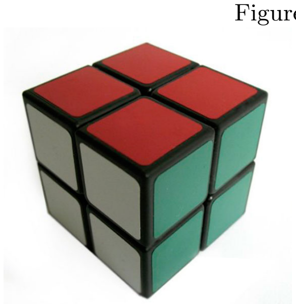
$$C_n(\Gamma) = \{c : V(\Gamma) \to [n] | c(W_1) \neq c(W_2), \forall (W_1, W_2) \in E(\Gamma)\}$$

denote the space of n-colorings of the graph Γ.

We regard $C_n(\Gamma)$ as a closed subspace of the Cantor set $[n]^{V(\Gamma)}$. If $deg(\Gamma) \leq k$, then $C_{k+1}(\Gamma) \neq \emptyset$.

A coloring $c \in C_n(\Gamma(\mathcal{X}))$ gives rise to a hierarchy of $\mathcal{X}$: cut along the hyperplanes colored 1, then the hyperplanes colored 2, ..., and finally the hyperplanes colored n.

Figure 21. A polyhedron with colored facets



What is left at the ends are stars of the vertices of $\mathcal{X}$, with residues of the colorings remaining on the boundary facets. We call these colored polyhedra (Figure 21).

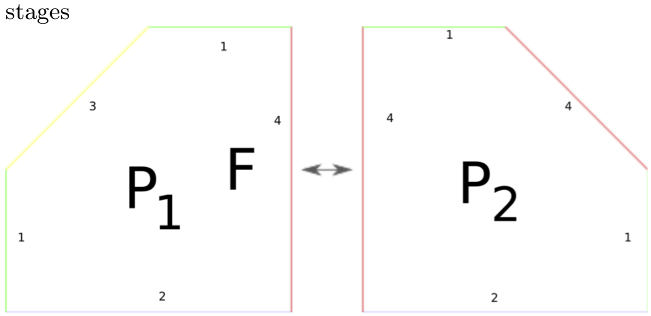
The idea of the proof is to "reverse-engineer" a hierarchy of a finite-sheeted cover of X, which is modeled on the hierarchy coming from a coloring of $\mathcal{X}$. We want to find a finite collection of colored polyhedra which is balanced, so that the number of colorings of a face is the same for the two polyhedra containing the face.

Then we may glue together polyhedra inductively, in order to reverse-engineer a hierarchy of a finite-sheeted cover, which is therefore virtually special by Wise (Figure 22).

7.1. Colorings of graphs.

I'll discuss a lemma which is used in the proof of Wise's conjecture.

Figure 22. Gluing polyhedra at the 4th stage of the hierarchy should preserve the lower stages



Let Γ be a graph of bounded valence $\leq k$, and let $\mathcal{G}$ be a group acting cocompactly on Γ.

Let $C_n(\Gamma)$ be the space of all colorings of Γ. Then $C_n(\Gamma)$ is a compact topological space, considered as a closed subspace of the Cantor set $[n]^\Gamma$.

Lemma 7.4. *There exists a probability measure μ on $C_{k+1}(\Gamma)$ which is $\mathcal{G}$-invariant.*

The proof of this lemma proceeds by coloring the vertices $V(\Gamma)$ randomly with n-colors, $n \geq k+1$. The probability that two endpoints of and edge $e \in E(\Gamma)$ have the same color is $1/n$. One can produce an $n-1$-coloring of the vertices, by sending each vertex colored n to the smallest color unused by its neighbors. By induction then, one produces a measure on $k+1$-colorings of $V(\Gamma)$ which have probability of coloring the endpoints of e the same color as $\leq 1/n$. Taking a weak-* limit of these measures, one obtains a $\mathcal{G}-invariant$ measure μ on $V(\Gamma)^{[k+1]}$ which is supported on the colorings of Γ.

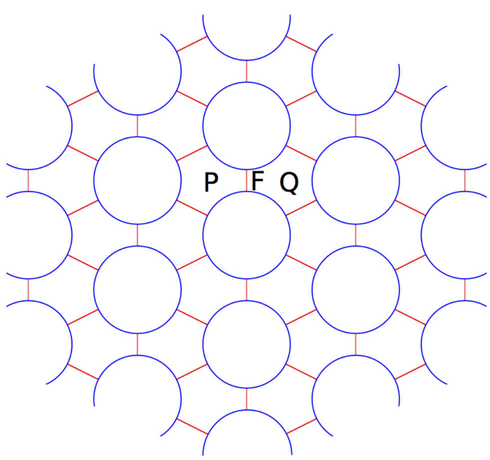
7.2. Colorings and hierarchies.

The probability measure is just an artifice to construct a solution to the *gluing equations*. We want to reverse engineer a hierarchy of a finite-sheeted cover. We have a finite (non-compact) hierarchy associated to the cover $\mathcal{X}$. The probability measure allows us to extract some finiteness associated to this hierarchy.

7.3. Polyhedra and facets.

Let $\mathcal{P}$ denote the stars of vertices of $\mathcal{X}$, which we will call *polyhedra*. Let $\mathcal{F}$ denote the facets of $\mathcal{X}$, which are dual to each edge of $\mathcal{X}$, and are the facets of the polyhedra $\mathcal{P}$. Each facet $F \in \mathcal{F}$ will be contained uniquely in two polyhedra $P, Q \in \mathcal{P}$, $P \cap Q = F$. There are 4 polygons in the example in Figure 19 up to the action of $\mathcal{G}$ (we won't draw P' and Q' which are duplicates of P and Q). As a concrete example, we take a covering space of X which kills the red curves, and kills the third power of the blue curves, giving a cover looking like Figure 23. Note that only half of the cover is drawn; the other half is obtained by doubling along the blue curves to get an infinite surface without boundary.

Figure 23. The cover $\mathcal{X}$ of the cube complex in Figure 19 and polyhedra and facet



7.4. Supercoloring. Each polyhedron and facet of $\mathcal{X}$ will correspond uniquely to one of X via the covering $\mathcal{X} \to X$.

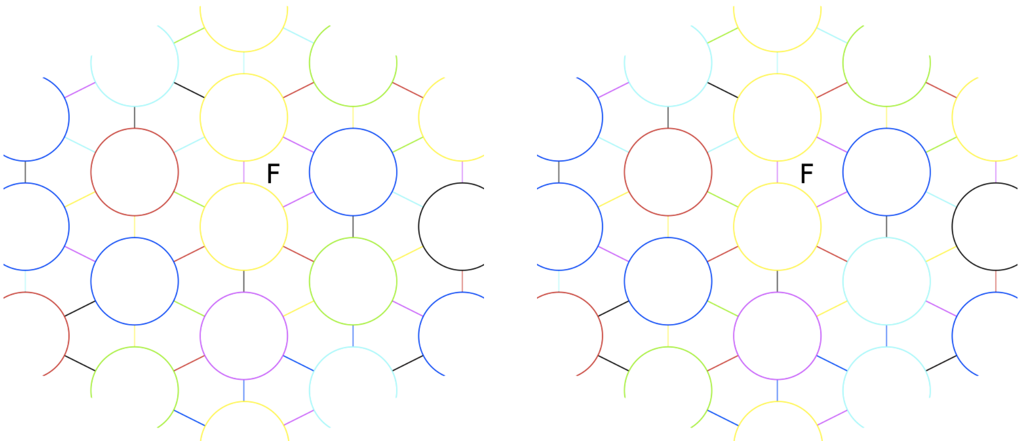
We refine the $k+1$-coloring of the hyperplanes $\mathcal{W}$ by the coloring of a neighborhood of size j in $\Gamma(\mathcal{X})$, where j is the color of a vertex, to get *supercolored* hyperplanes. The facets $F \in \mathcal{F}$ get supercolored by their corresponding hyperplanes, and polyhedra will be supercolored by their facets.

7.5. Polyhedral gluing equations. The variables for the gluing equations will be super colored polyhedra, and the gluing equations will say that for a given super colored facet F, the super colorings of P which induce the same super coloring of F must equal the super colorings of Q which induce the super coloring of F. We require that the variables are $\mathcal{G}$-invariant, in which case they are determined by finitely many variables corresponding to the polyhedra of X (or $\mathcal{G}$-orbits of super colored polyhedra of $\mathcal{X}$).

The $\mathcal{G}$-invariant measure μ gives a solution to the gluing equations with non-negative weights. Then we can get an integral solution to the gluing equations with non-negative weights, since the equations are linear with integral coefficients. We take the integral solution to the polyhedral gluing equations, and use them to glue up a finite-sheeted cover of X, which is “modeled” on the hierarchies associated to colorings of $\mathcal{X}$.

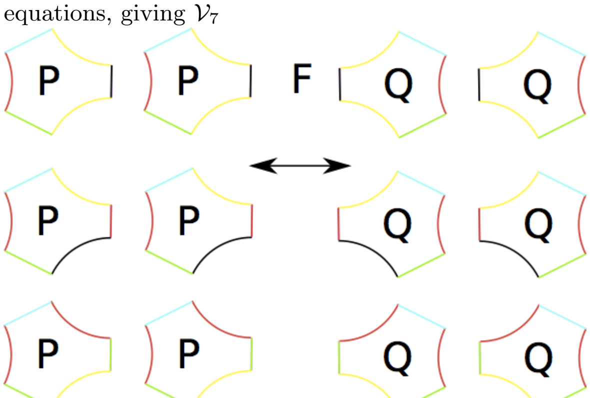
7.6. Gluing up the hierarchy. We construct a sequence of (usually disconnected) finite cube complexes $\mathcal{V}_j$, $k+1 \geq j \geq 0$, with boundary pattern $\{\partial_1(\mathcal{V}_j), \ldots, \partial_j(\mathcal{V}_j)\}$ determined by the unpaired faces colored j. The final stage $\mathcal{V}_0$ will be a finite-sheeted cover of X. The first stage $\mathcal{V}_{k+1}$ is obtained by taking a number of copies of each supercolored polyhedron determined by the integral solution to the gluing equations. In our example, $k=6$, so the first stage is $\mathcal{V}_7$ (Figure

Figure 24. The face F has different supercolorings, even though the facet is colored the same in both colorings



25). If we glued the faces of the polyhedra $\mathcal{V}_{k+1}$ together preserving colors, then we would obtain a finite-sheeted branched cover of X. So we have to be careful at each stage that the gluing extends to an unbranched covering space.

Figure 25. Collection of supercolored polyhedra determined by the solution to the gluing equations, giving $\mathcal{V}_7$



The next stage of the hierarchy $\mathcal{V}_k$ is obtained from $\mathcal{V}_{k+1}$ by gluing the faces labeled $k+1$ in pairs along matching supercolored faces (in our example, $k+1 = 7$ is represented by black, obtaining $\mathcal{V}_6$, Figure 26).

We glue $\mathcal{V}_5$ from a cover $\tilde{\mathcal{V}}_6$ of $\mathcal{V}_6$ by gluing the boundary pattern $\partial_6\mathcal{V}_6$ (which in our example is colored yellow Figure 27):

The supercoloring guarantees that the two sides of $\partial_6\mathcal{V}_6$ have consistently su-

Figure 26. Gluing $\mathcal{V}_7$ to get $\mathcal{V}_6$

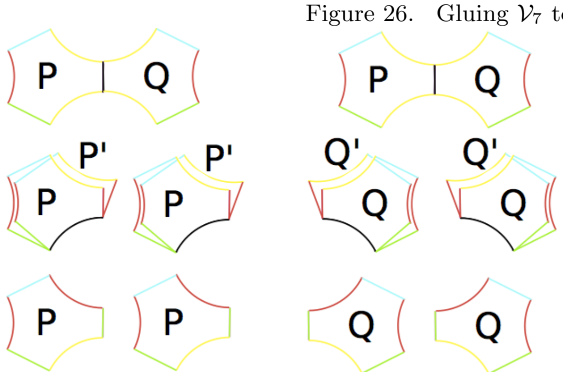
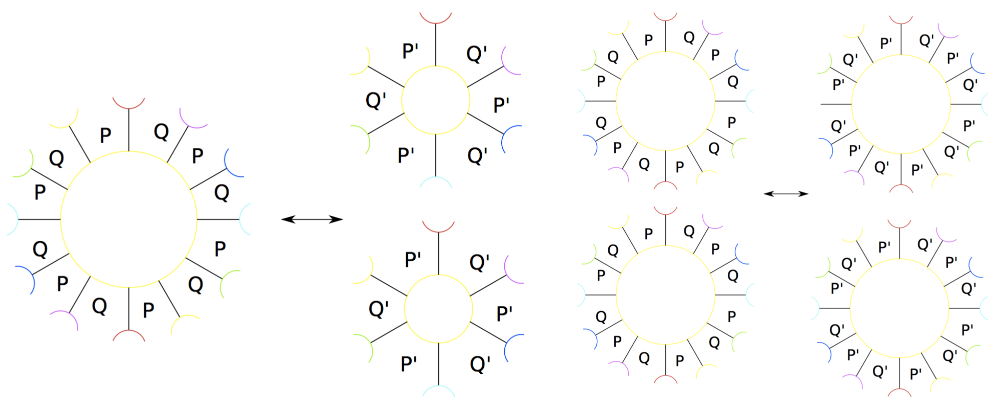


Figure 27. Taking a cover $\tilde{\mathcal{V}}_6$ of $\mathcal{V}_6$ to be able to glue to get $\mathcal{V}_5$



(a) $\mathcal{V}_6$ (b) $\tilde{\mathcal{V}}_6$

percolored hyperplanes, and therefore is a finite-sheeted cover of the hyperplane in a representative coloring of $\mathcal{X}$ (Figure 27 (a)). The MSQT allows us to pass to a finite-sheeted cover $\tilde{\mathcal{V}}_6$ in which both sides of $\partial_6\tilde{\mathcal{V}}_6$ match by an isometry (Figure 27 (b)).

We obtain $\mathcal{V}_i$ from $\mathcal{V}_{i+1}$ by finding a covering space $\tilde{\mathcal{V}}_{i+1} \to \mathcal{V}_{i+1}$ in which the boundary pattern $\partial_{i+1}\tilde{\mathcal{V}}_{i+1}$ may be matched up in pairs which reverse the coorientations and preserve super colorings. Constructing the cover $\tilde{\mathcal{V}}_{i+1}$ requires another application of Wise's MSQT.

The cube complex $\mathcal{V}_0$ will have no boundary pattern, and thus will give a finite-sheeted covering space $\mathcal{V}_0 \to X$ and which has by construction has embedded acylindrical hyperplanes, and therefore a malnormal hierarchy.

One more application of Wise's theorem ($\mathcal{MQH} \implies$ virtually special) gives a cover $\tilde{\mathcal{V}}_0 \to X$ which is special.

8. 3-manifold applications

8.1. Non-positive curvature. We state a result that combines the statements of theorems of Liu and Przytycki-Wise:

Theorem 8.1. *[86, Theorem 1.1] [105, Corollary 1.4] Let M be an aspherical compact 3-manifold. The following are equivalent:*

1. *M admits a complete metric of non-positive curvature*
2. *M is virtually homotopic to a special cube complex*
3. *$\pi_1(M)$ virtually embeds in a right-angled Artin group*
4. *$\pi_1(M)$ is virtually RFRS*

In particular, such manifolds are virtually fibered.

The manifolds which do not admit a metric of non-positive curvature are graph manifolds, and have been characterized by Svetlov in terms of the BKN equations [115].

A corollary of this result is that if M admits a non-positively curved metric, then $\pi_1(M)$ is linear (in fact, embeds in $GL(n,\mathbb{Z})$). It is still unresolved whether graph manifold groups are linear. It would be remarkable if there are examples of fibered graph manifolds with non-linear fundamental group, since it would imply the existence of non-linear mapping class groups.

8.2. Virtual torsion. Let K be a finitely generated abelian group.

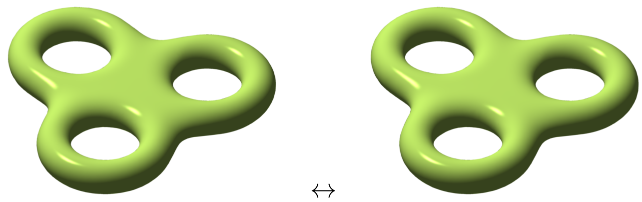
Theorem 8.2. *[113, Sun 2013] Given M a closed hyperbolic 3-manifold, there is a finite-sheeted cover $\tilde{M} \to M$ such that $H_1(M) \cong K \oplus L$.*

For each summand $\mathbb{Z}/N\mathbb{Z}$ of K, Sun constructs an immersed complex $C_N \to M$ which has a surface with one boundary component which wraps N times around a loop, and such that $\pi_1(C_N) \to \pi_1(M)$ is an injection, in fact with quasiconvex image. Take such a complex for each cyclic summand of K, and immerse a wedge of these complexes in M to get a quasiconvex immersion of a complex $C \to M$ such that $\pi_1(C) < \pi_1(M)$ is quasiconvex. We also have by construction $H_1(C;\mathbb{Z}) \cong K$. By the virtual retract property, there is a cover $\tilde{M} \to M$ such that there is a retract $r : \pi_1(\tilde{M}) \to \pi_1(C)$. Then we have a retract $r_* : H_1(\tilde{M}) \to H_1(C)$. Therefore, we have $H_1(\tilde{M}) \cong H_1(C) \oplus ker(r_*) = K \oplus L$, for $L = ker(r_*)$.

Theorem 8.3. *[114, Sun 2014] Let M be a closed hyperbolic 3-manifold. For any closed manifold N, there is a finite cover $\tilde{M} \to M$ such that there is a degree 2 map $\rho : \tilde{M} \to N$.*

Since the map $H^3(N;\mathbb{Z}/p\mathbb{Z}) \to H^3(\tilde{M};\mathbb{Z}/\mathbb{Z})$ is an isomorphism for p odd, there is an embedding of cohomology rings $H^*(N;\mathbb{Z}/p\mathbb{Z}) \to H^*(\tilde{M};\mathbb{Z}/p\mathbb{Z})$. Thus, not only can one achieve arbitrary torsion in covers of a hyperbolic 3-manifold, but one can also embed any cohomology ring of a 3-manifold, at least with odd order coefficients (one may also use rational coefficients).

Figure 28. A homeomorphism defining a Heegaard splitting



8.3. Heegaard gradient.

For M a closed 3-manifold, the **Heegaard genus** $g(M)$ is the minimal genus of a surface $\Sigma_g \subset M$ such that Σ_g bounds handlebodies to each side (Σ_g is a **Heegaard surface** Figure 28). The **Heegaard gradient** of M is

$$\nabla g(M) = \inf_{\tilde{M}\to M \ finite} \frac{2g(\tilde{M})-2}{[\pi_1 M : \pi_1 \tilde{M}]}$$

This notion was introduced by Lackenby to probe the virtual Haken conjecture [78].

If M is fibered, then it is easy to see that $\nabla g(M) = 0$.

Conjecture 8.4. (Lackenby [78]) Let M be a closed hyperbolic 3-manifold. M is virtually fibered if and only if $\nabla g(M) = 0$.

This conjecture now follows (essentially trivially) from the virtual fibering conjecture, and therefore hyperbolic 3-manifolds have $\nabla g(M) = 0$. Note that Ichihara has shown that Seifert-fibered 3-manifolds with infinite fundamental group have zero Heegaard gradient, even though some of them are not virtually fibered. It remains to compute the Heegaard gradients of graph manifolds which are not virtually fibered.

Theorem 8.5. *A closed orientable 3-manifold M has $\nabla g(M) \leq 0$ if and only if it is prime or $M \cong \mathbb{RP}^3 \# \mathbb{RP}^3$.*

Proof. First, note that if M is not prime or $\mathbb{RP}^3 \# \mathbb{RP}^3$, then $\nabla g(M) > 0$. The sphere decomposition of M gives a graph-of-groups decomposition of $\pi_1 M$ with trivial edge groups. After passing to a finite-sheeted cover, one may assume that the vertices of this graph all have degree ≥ 3. Then the corank of $\pi_1 M$ is > 1 (the **corank** is the maximal rank free group surjected by $\pi_1(M)$). As one passes to further finite-sheeted covers, the corank grows at least linearly with the index, and therefore the corank gradient is > 0, a fortiori the rank gradient and Heegaard gradient.

If $|\pi_1(M)| < \infty$, then $\nabla g(M) < 0$ by the Poincaré conjecture, and if $|\pi_1(M)| \cong \mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, then $\nabla g(M) = 0$.

Now, suppose M is aspherical. If M has non-zero Gromov norm, then M virtually fibers, and therefore $\nabla g(M) = 0$ [105]. If M has zero Gromov norm, then M is a graph manifold.

If M is Seifert-fibered (with infinite fundamental group), then this was proved by Ichihara [71]. It is easy to check that this holds for graph manifolds with non-trivial JSJ decomposition as well. There is a finite-sheeted cover in which each Seifert piece is homeomorphic to $\Sigma \times S^1$, for some surface with boundary Σ. The Heegaard genus of each piece is $b_1(\Sigma)+1$. By passing to a further cover, we may assume that the JSJ decomposition is bipartite, so that $M = M_1 \cup_T M_2$, where $M_i \cong \Sigma_i \times S^1$ (Σ_i may be disconnected), and $T = \partial M_1 = \partial M_2$ is the union of JSJ tori. Each M_i has a Heegaard splitting of genus $b_1(M_i)$, in $M_i = H_i \cup C_i$, where H_i is a union of handlebodies of genus $b_1(M_i)$, and $T = \partial M_i \subset \partial C_i$. We may construct a Morse function on M, which has T as a level set, and induces a perfect Morse function M_i, which is standard on H_i and C_i (although the restriction to M_2 will have the indices flipped). The index 0 critical points lie in H_1, so that there are $b_0(M_1)$ index 0 critical points, and similarly the index 3 critical points lie in H_2, so there are $b_0(M_2)$ of them. There are $b_1(H_1) = b_1(M_1)$ critical points of index one in H_1, and there are $b_2(M_2) = b_1(M_2) - b_0(M_2)$ critical points of index one in C_2. Similarly, there are $b_1(H_2) = b_1(M_2)$ critical points of index two in H_2, and $b_2(M_1) = b_1(M_1) - b_0(M_1)$ critical points of index two in C_1. Now, detelescope the Morse function on M to get a Heegaard splitting of M, then cancel all but one of the index 0 critical points with index 1 critical points. This gives a Morse function with $b_1(H_1) + b_1(M_2) - b_0(M_2) - b_0(M_1) = b_2(M_1) + b_2(M_2) = b_1(\Sigma_1) + b_1(\Sigma_2)$ index one critical points.

Now we observe that by passing to covering spaces $\tilde{M} \to M$, we may make the ratio $(b_1(\tilde{\Sigma}_1) + b_1(\tilde{\Sigma}_2))/[\tilde{M} : M]$ arbitrarily close to zero, by unwrapping the S^1 direction of M_1 and M_2 an arbitrarily large amount. This shows that the Heegaard gradient is zero. □

Definition 8.6. Let G be a group, then $d(G)$ is the minimal number of generators needed to generate G (if G is not finitely generated, set $d(G) = \infty$).

Now, suppose G is a residually finite group. Define

$$\nabla d(G) = \inf_{\tilde{G} < G, [G:\tilde{G}] < \infty} \frac{d(\tilde{G}) - 1}{[G : \tilde{G}]}.$$

If (G_n) is a chain of subgroups $G_{n+1} < G_n < G$ with $[G : G_n] < \infty$, define

$$\nabla d(G, (G_n)) = \lim_{n\to\infty} \frac{d(G_n) - 1}{[G : G_n]}.$$

Clearly $\nabla d(G) \leq \nabla d(G, (G_n))$.

Let M be a closed aspherical 3-manifold with $\pi_1(M) = G$. Then $2d(G) \leq g(M)$, and therefore $\nabla d(G) = \frac{1}{2}\nabla g(M) = 0$. It is known that there are manifolds M with $d(G) < \frac{1}{2} g(M)$ [22, 85]. If M is hyperbolic, and $G_n < G$ are congruence subgroups, then it is shown that $\nabla g(M, G_n) > 0$ [78, 88]. As observed by Abert-Nikolov, the fixed price conjecture of Gaboriau would imply that $\nabla d(\pi_1 M, G_n) = 0$ for any cofinal chain (G_n).

Question: For M a closed 3-manifold with $\pi_1 M = G$, what is $\inf_{[M:N]<\infty} d(\pi_1 N)/g(N)$?

For further properties of 3-manifold groups following from , we refer to the comprehensive survey [11, Section 6].

9. Cubulated groups

Theorem 9.1. *[92, Markovic 2012] Let Γ be a word-hyperbolic group, such that $\partial_\infty\Gamma \cong S^2$ (and Γ acts effectively on $\partial_\infty\Gamma$). Suppose moreover that Γ is cubulated. Then Γ is isomorphic to a Kleinian group. In particular, if Γ is torsion-free, then $\Gamma = \pi_1(M)$ for some closed hyperbolic 3-manifold.*

This gives a possible approach to Cannon's conjecture, which is that Γ is a Kleinian group. One could try to carry out the technique of Kahn-Markovic to try to find quasiconvex subgroups with limit sets circles in $\partial_\infty\Gamma$ which satisfy Bergeron-Wise's condition, that one can separate any pair of points in $\partial_\infty\Gamma$ by a circle limit set of a surface subgroup.

We remark that there are many other classes of cubulated hyperbolic groups to which Theorem 6.6 applies: $C'(\frac{1}{6})$ groups [124], random groups at density $< \frac{1}{6}$ [101], certain ascending HNN extensions of free groups [29, 60], and isometry groups of certain polygonal complexes [43, 47].

10. Group theoretic applications

We point out a minor observation regarding Haglund-Wise's theorem [62]:

Theorem 10.1. *Let G act properly cocompactly and virtually specially on a cube complex X. Then G embeds in a finite extension of a RAAG.*

The point is that there is a normal subgroup $G' \lhd G$ such that X/G' is special. The embedding $X/G' \to S_{\Gamma(X/G')}$ is functorial, in that combinatorial automorphisms of X/G' extend to $S_{\Gamma(X/G')}$. Thus, G embeds in an extension of $A_{\Gamma(X/G')}$ by G/G'. Thus, all hyperbolic 3-manifold groups embed in finite extensions of RAAGs. This observation may have importance, for example, in understanding the representations of 3-manifold groups, by examining the representations of finite extensions of RAAGs.

We point out another consequence of virtual specialness. Given a RF group G, let $\hat{G}$ denote its profinite completion.

Definition 10.2. A group G is good if for every finite $\hat{G}$-module M, there is an isomorphism $H^*(\hat{G}, M) \cong H^*(G, M)$.

Theorem 10.3. *Let G be a virtually compact special group. Then G is good.*

Proof. This follows by induction from [57, Proposition 3.6]. If G is virtually compact special, then it has a finite-index subgroup which admits a quasi-convex hierarchy. Then $G = A *_C B$, where A, B, C are virtually compact special. By induction, A, B, C are good groups. Also, by [63], the groups A, B, C are virtual retracts, and therefore are efficient. So by [57, Proposition 3.6], we conclude that G is good. □

Remark: We cannot apply directly [57, Proposition 3.9], since we don't know that G is subgroup separable, only that quasiconvex subgroups are separable.

In general, cubulated hyperbolic groups are not LERF, for example by Rips' construction [107, 124]. However, quasiconvex subgroups are separable. So it is natural to ask for which hyperbolic groups are finitely generated subgroups quasiconvex? This is a strong form of coherence.

Theorem 10.4. *Negatively curve square complex groups are LERF.*

A negatively curved square complex has vertex links graphs of girth ≥ 5, so that it admits a CAT(-1) metric making each square a hyperbolic square with angles $2\pi/5$. This follows from a result of McCammond-Wise that negatively curved square complexes are locally convex [98].

11. Open questions

1. (Long-Reid) Can two Kleinian groups which are non-isomorphic have the same profinite completion?

   Remark: This is equivalent to the question, given two hyperbolic 3-manifold groups, do they have the same collection of finite quotients?

2. Are compact 3-manifold fundamental groups linear?

   Remark: The only aspherical case left is graph manifolds which don't admit a non-positively curved metric by Theorem 8.1.

3. Is there an algorithm to detect if a compact cube complex is virtually special?

4. Find a bound on the index of a cover of an aspherical 3-manifold which is Haken. The bound should be some computable function of some complexity of the 3-manifold, such as the minimal number of tetrahedra of a triangulation. In principle, there is an algorithm which will find a Haken cover. The most practical approach is likely to enumerate homomorphisms $\rho : \pi_1(M) \to K$, K a finite group, and compute $H_1(\pi_1(M); \mathbb{Q}[K])$, which is the homology of the covering space corresponding to $ker(\rho)$ [44].

5. Let M be a 3-manifold with $rank(H_1(M; \mathbb{F}_p)) \geq 4$. Does M admit a regular p-cover $\tilde{M}$ with $b_1(\tilde{M}) > 0$? If this were true, it might yield a more practical approach to finding Haken covers [80].

6. For any two hyperbolic 3-manifolds M_1, M_2, are there fibered covers $M_i' \to M_i$ such that there is a non-zero degree map $M_1' \to M_2'$ which preserves the fibering?

7. Do closed hyperbolic 3-manifolds contain immersed quasi-fuchsian surfaces of odd Euler characteristic?

8. [100, Niblo-Wise] Which 3-manifold groups are LERF? No Seifert-Seifert gluings in JSJ?

9. Consider a hyperbolic group G which acts properly on a cube complex with finitely many orbits of hyperplanes, but not necessarily cocompactly. Is G virtually special?

10. Which knot groups are RFRS?

11. Are braid groups B_n RFRS? **Remark:** Mapping class groups are not virtually RFRS in general (cf. [86, Liu]).

12. Does a finite volume hyperbolic 3-manifold M admit a cover which fibers over S^1 with orientable foliation of the pseudo-Anosov map?

13. For M a finite-volume hyperbolic 3-manifold, $\Gamma = \pi_1(M)$, does $rank(\Gamma) = rank(\hat{\Gamma}) = max\{rank(\Gamma/N)|N \lhd \Gamma, [\Gamma : N] < \infty\}$? Note that M has a finite-sheeted cover $\tilde{M} \to M$ which has this property, in fact such that $rank(\pi_1(\tilde{M})) = rank H_1(\tilde{M}; \mathbb{Z}/2\mathbb{Z})$, since a fibered manifold always has a finite-sheeted cover with this property.

14. Is there a strengthening of the Malnormal Special Quotient Theorem? Let G be hyperbolic and cubulated and (G, P) be relatively hyperbolic. Are there finitely many elements that we may exclude in P so that any Dehn filling which avoids these elements is cubulated? This would be a strengthening of the MSQT (and there is an obvious generalization to multiple peripheral subgroups).

15. Which cocompact lattices in hyperbolic buildings are cubulated?

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