Upper and Lower Bounds
on the Norms of Functions of Matrices

Given an n by n matrix A, we look for a set S in the complex plane and positive constants m and M such that for all polynomials (or analytic functions) p, the inequalities

$$m \cdot \inf\{\|f\|_{\mathcal{L}^\infty(S)} : \ f(A) = p(A)\} \leq \|p(A)\| \leq M \cdot \inf\{\|f\|_{\mathcal{L}^\infty(S)} : \ f(A) = p(A)\}$$

hold, where $\|\cdot\|$ denotes the operator 2-norm. We show that for 2 by 2 matrices, if S is the field of values, then one can take $m = 1$ and $M = 2$. We show that for a perturbed Jordan block – a matrix A that is an n by n Jordan block with eigenvalue 0 except that its $(n,1)$-entry is $\nu \in (0,1)$ – if S is the unit disk, then $m = M = 1$. More generally, we show that if A is a companion matrix whose eigenvalues lie in the open unit disk $\mathcal{D}$, then $m = 1$ if $S = \mathcal{D}$. We consider the relation of this work to the conjecture of Crouzeix that if S is the field of values of A then the constant M can be taken to be 2.