Eigenvalues of a Perturbed Hermitian Matrix

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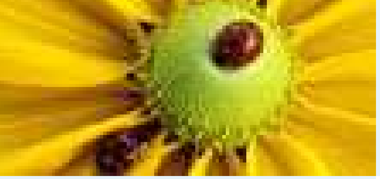
joint work: Chi-Kwong Li, College of William and Mary



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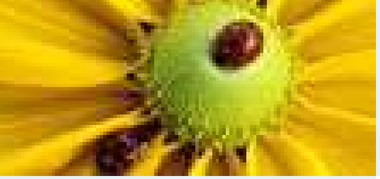
from its submatrices H_1 and H_2.



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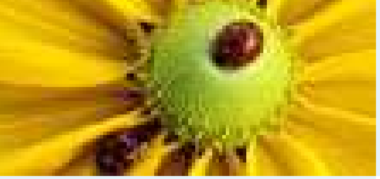

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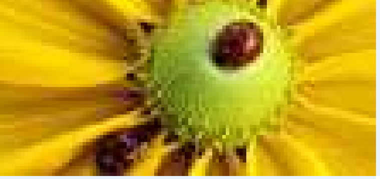
B. N. Parlett, ***The Symmetric Eigenvalue Problem***, Chapter 10: a comprehensive collection of results.



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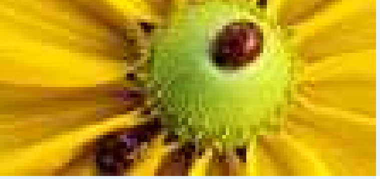

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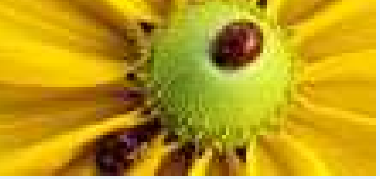
(Multiple eigenvalues appear as many times as their algebraic multiplicities.)



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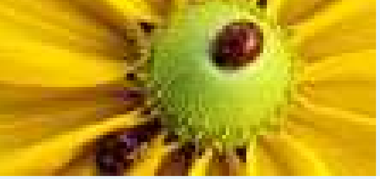
▷ Eigenvalue gaps:

$$\begin{aligned} \eta_i &\stackrel{\text{def}}{=} \text{dist}(\lambda_i(\widetilde{A}), \text{eig}(H_j)) \quad \text{for } \lambda_i(\widetilde{A}) \notin \text{eig}(H_j), \\ \delta &\stackrel{\text{def}}{=} \text{dist}(\text{eig}(H_1), \text{eig}(H_2)) \equiv \min_i \eta_i. \end{aligned}$$

Some Known Results

▷ Cauchy interlacing properties: give inclusion intervals using some in $\text{eig}(H_i)$ to contain some in $\text{eig}(A)$.



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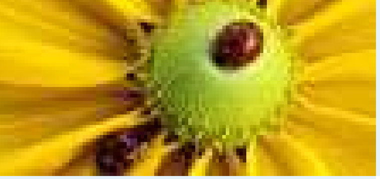

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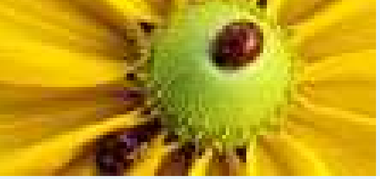

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▷ Kahan, Parlett, Demmel: tri-diagonal matrices – when to neglect an off-diagonal entry; don't use $\text{eig}(H_i)$ directly.

Some Known Results (cont'ed)

$$|\lambda_i(A) - \widetilde{\lambda}_i(\widetilde{A})| \leq \|E\|, \quad (1)$$

$$|\lambda_i(A) - \widetilde{\lambda}_i(\widetilde{A})| \leq \frac{\|E\|^2}{\delta}. \quad (2)$$

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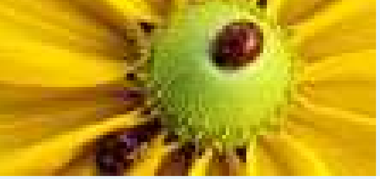

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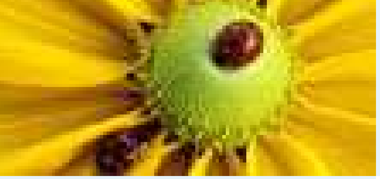
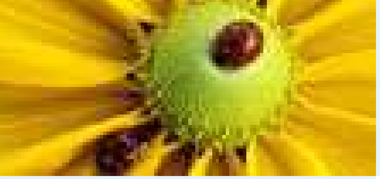
Good enough for any practical purpose. ***Not that mathematically elegant, however!***



2-by-2 Example

$A = \begin{pmatrix} \alpha & \varepsilon \\ \varepsilon & \beta \end{pmatrix}$. Delete ε to get $\widetilde{A} = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$.

Assume $\alpha > \beta$. Tiny $\varepsilon \Rightarrow \text{eig}(A) \approx \{\alpha, \beta\}$.



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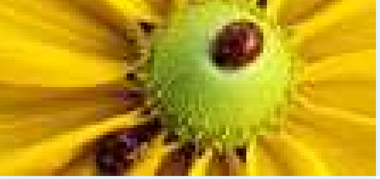

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Errors:

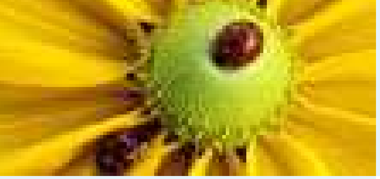
$$0 < \left\{ \begin{array}{c} \lambda_+ - \alpha \\ \beta - \lambda_- \end{array} \right\} = \frac{2\varepsilon^2}{(\alpha-\beta) + \sqrt{(\alpha-\beta)^2 + 4\varepsilon^2}}$$

$$\begin{cases} \le |\varepsilon| & \text{always,} \\ \to |\varepsilon| & \text{as } \alpha \to \beta^+, \\ \le \varepsilon^2/(\alpha-\beta). \end{cases}$$

New Result

$$|\lambda_i(A) - \lambda_i(\widetilde{A})| \leq \frac{2\|E\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|E\|^2}} \tag{3}$$

$$\leq \frac{2\|E\|^2}{\delta + \sqrt{\delta^2 + 4\|E\|^2}}. \tag{4}$$



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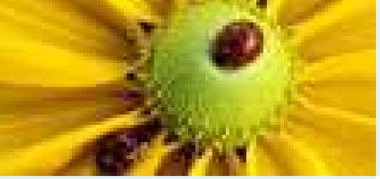

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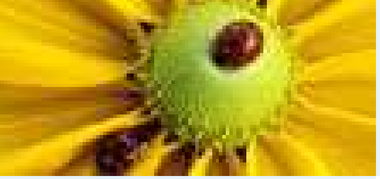

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Otherwise replace A by $(U_1 \oplus U_2)^* A (U_1 \oplus U_2)$.

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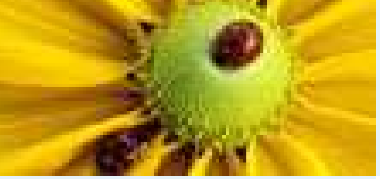
▷ $N = 2$: done by the example. Assume true for $N - 1$.

Outline of Proof (cont'ed)

▷ $\lambda_i(A) - \lambda_i(\widetilde{A})$ for $i = 1$, i.e., for the largest eigenvalues.



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$$A = \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \xrightarrow{\text{sym. permute}} \begin{pmatrix} H_2 & E \\ E^* & H_1 \end{pmatrix}.$$

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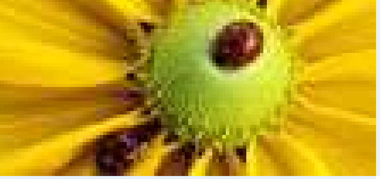
Then $\lambda_1(A) \geq e_1^* A e_1 = \lambda_1(\widetilde{A})$. No proof needed if $\lambda_1(A) = \lambda_1(\widetilde{A})$. Assume $\lambda_1(A) > \lambda_1(\widetilde{A})$.



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Set $X = \begin{pmatrix} I & 0 \\ -(H_2 - \lambda_1(A)I)^{-1}E & I \end{pmatrix}$, and then

$$X^*(A - \lambda_1(A)I)X =$$

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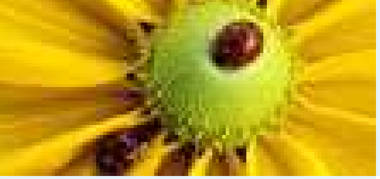
$A - \lambda_1(A)I$ and $X^*(A - \lambda_1(A)I)X$ have the same inertia $\Rightarrow$ $H_1 - \lambda_1(A)I - E^*(H_2 - \lambda_1(A)I)^{-1}E$ has zero as its largest eigenvalue.



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Consider $H_1 - \lambda_1(A)I$ and $H_1 - \lambda_1(A)I - E^*(H_2 - \lambda_1(A)I)^{-1}E$. Largest eigenvalue of $H_1 - \lambda_1(A)I$ is $\lambda_1(\widetilde{A}) - \lambda_1(A) < 0$. Thus

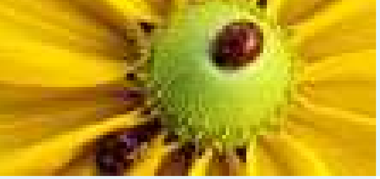
$$|[\lambda_1(\widetilde{A}) - \lambda_1(A)] - 0| \leq \|E^*(H_2 - \lambda_1(A)I)^{-1}E\|.$$

Outline of Proof (cont'ed)

Let $\gamma_1 \overset{\text{def}}{=} \lambda_1(A) - \lambda_1(\widetilde{A})$ and note

$$\begin{aligned}
\|E^*(H_2 - \lambda_1(A)I)^{-1}E\| &\le \frac{\|E\|^2}{\lambda_1(A) - \lambda_1(H_2)} \\
&= \frac{\|E\|^2}{\lambda_1(A) - \lambda_1(\widetilde{A}) + \lambda_1(H_1) - \lambda_1(H_2)} \\
&= \frac{\|E\|^2}{\gamma_1 + \eta_1}
\end{aligned}$$

to get $\gamma_1 \le \frac{\|E\|^2}{\gamma_1 + \eta_1}$ which implies $\gamma_1 \le \frac{2\|E\|^2}{\eta_1 + \sqrt{\eta_1^2 + 4\|E\|^2}}$.



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▷ Done also with $i = N$ (for smallest eigenvalues) by working with $-A$ and $-\widetilde{A}$.

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Can Assume $\lambda_i(\widetilde{A}) > \lambda_i(A)$. Otherwise, replace $(A, \widetilde{A}, i)$ by $(-A, -\widetilde{A}, N - i + 1)$.

Outline of Proof (cont'ed)

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Delete corresponding row and column of A containing $\lambda_N(\widetilde{A})$ to get

$$A = \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \xrightarrow{\text{del. row, col.}} \widehat{A} = \begin{pmatrix} \widehat{H}_1 & \widehat{E}^* \\ \widehat{E} & \widehat{H}_2 \end{pmatrix}.$$

Cauchy interlacing property gives $\lambda_i(A) \geq \lambda_i(\widehat{A})$. Thus

$$\lambda_i(\widetilde{A}) - \lambda_i(A) \leq \lambda_i(\widetilde{A}) - \lambda_i(\widehat{A}). \tag{5}$$

$\lambda_i(\widetilde{A})$ is the ith largest diagonal entries in $\widehat{A}$. Define $\widehat{\eta}_i = \text{dist}(\lambda_i(\widetilde{A}), \text{eig}(\widehat{H}_j))$ for $\lambda_i(\widetilde{A}) \notin \text{eig}(\widehat{H}_j)$. Then

$$\widehat{\eta}_i \geq \eta_i$$

because $\widehat{H}_j$ may have one fewer diagonal entries than H_j.

Outline of Proof (cont′ed)

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$$\widehat{\eta}_i \geq \eta_i$$

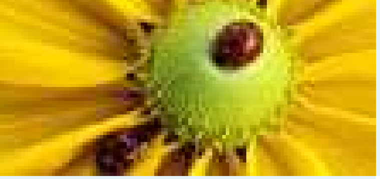
because $\widehat{H}_j$ may have one fewer diagonal entries than H_j.

$\widehat{A}$ is $(N-1)$-by-$(N-1)$. Conclusion holds by assumption. Thus

Outline of Proof (cont'ed)

$$
\begin{aligned}
|\lambda_i(A) - \lambda_i(\widetilde{A})| &= \lambda_i(\widetilde{A}) - \lambda_i(A) && \text{because } \lambda_i(\widetilde{A}) > \lambda_i(A) \\
&\leq \lambda_i(\widetilde{A}) - \lambda_i(\widehat{A}) && \text{by (5)} \\
&\leq \frac{2\|\widehat{E}\|^2}{\widehat{\eta}_i + \sqrt{\widehat{\eta}_i^2 + 4\|\widehat{E}\|^2}} && \text{by induction assump.} \\
&\leq \frac{2\|\widehat{E}\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|\widehat{E}\|^2}} && \text{because } \widehat{\eta}_i \geq \eta_i \\
&= \frac{1}{2}\sqrt{\eta_i^2 + 4\|\widehat{E}\|^2} - \eta_i && \\
&\leq \frac{1}{2}\sqrt{\eta_i^2 + 4\|E\|^2} - \eta_i && \text{because } \|\widehat{E}\| \leq \|E\| \\
&= \frac{2\|E\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|E\|^2}} &&
\end{aligned}
$$

.



Singular Values

- Eigenvalues of a Perturbed Hermitian Matrix

The Problem

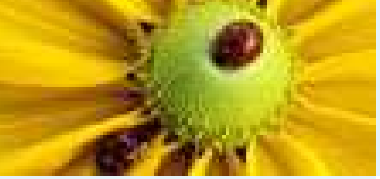
New Result

Singular Values
- **Singular Values**
- **Singular Values** (`cont'd`)

▷ For convenience, m-by-n X has singular values

$$\sigma_1(X) \geq \sigma_2(X) \geq \cdots \geq \sigma_{\max\{m,n\}}(X)$$

Last $\max\{m,n\} - \min\{m,n\}$ are known to be zeros always.



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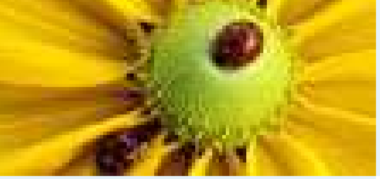
▷ sv$(X) = \{\sigma_i(X)\}$.



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▷ $\text{sv}(X) = \{\sigma_i(X)\}$.

▷ Let

$$B = \begin{pmatrix} G_1 & E_1 \\ E_2 & G_2 \end{pmatrix}, \; \widetilde{B} = \begin{pmatrix} G_1 & \\ & G_2 \end{pmatrix}.$$

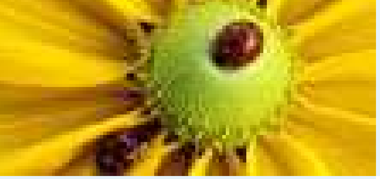
Clearly $\text{sv}(\widetilde{B}) = \text{sv}(G_1) \bigcup \text{sv}(G_2)$. Define

$$\begin{aligned} \eta_i &= \text{dist}(\sigma_i(\widetilde{B}), \text{sv}(G_j)) \text{ for } \sigma_i(\widetilde{B}) \notin \text{sv}(G_j), \\ \delta &= \min\{\eta_i\}. \end{aligned}$$

Singular Values (cont'd)

▷ ***Well-Known:***

$$|\sigma_i(B) - \sigma_i(\widetilde{B})| \leq \varepsilon.$$



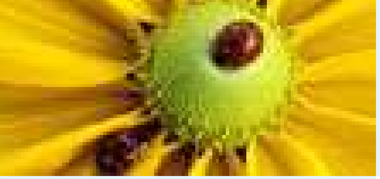
Singular Values (cont'd)

▷ ***Well-Known:***

$$|\sigma_i(B) - \sigma_i(\widetilde{B})| \leq \varepsilon.$$

▷ ***New Result:***

$$\begin{aligned} |\sigma_i(B) - \sigma_i(\widetilde{B})| &\leq \frac{2\varepsilon^2}{\eta_i + \sqrt{\eta_i^2 + 4\varepsilon^2}} \\ &\leq \frac{2\varepsilon^2}{\eta + \sqrt{\eta^2 + 4\varepsilon^2}}. \end{aligned}$$



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Proof: Work with Hermitian matrices

$$\begin{pmatrix} & B \\ B^* & \end{pmatrix}, \quad \begin{pmatrix} & \widetilde{B} \\ \widetilde{B}^* & \end{pmatrix}.$$