Recollections
(designed to entertain or enlighten)

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1959.

Take home final exam given by George Forsythe (Math 237A)

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Question 1. To how many decimal digits is π known?

Is it a good question?

My Epiphany (1961)

Find the eigenvector of W_{21}^{-} corresponding to the largest eigenvalue

$$10.746\ldots$$

$$W_{21}^{-} = \begin{bmatrix} 10 & 1 & & & & & & & & & \\ 1 & 9 & 1 & & & & & & & & \\ & 1 & 8 & 1 & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & \\ & & & 1 & 0 & 1 & & & & & \\ & & & & 1 & -1 & 1 & & & & \\ & & & & & 1 & -2 & 1 & & & \\ & & & & & & \ddots & \ddots & \ddots & & \\ & & & & & & & 1 & -9 & 1 \\ & & & & & & & & 1 & -10 \end{bmatrix}$$

Givens method

Disaster!

Get wrong eigenvector!

Get same (bad) output when Givens is executed in exact arithmetic.

[See Table 9 on page 321 of AEP]

Why?

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[See Table 9 on page 321 of AEP]

Why?

Roundoff is not the only source of error! There is

FINITE REPRESENTATION!

Cleve Moler's assessment of my [1973] manuscript on

The Unsymmetric Eigenvalue Problem

lead to my [1980] book on

The Symmetric Eigenvalue Problem

Ph.D. dissertations unpublished

Year	*Present Position*	*Present Position*
1983	K. C. Ng	SUN Microsystems, Inc.
1988	J. Le	EEG, San Francisco
1988	J. Li	Motorola, Austin
1991	Z. Liu	SUN Microsystems, Inc.
1991	Y. S. Feng	HP
1992	T. T. Lu	National SunYatSen University Kaoshing, Taiwan
1993	David Day	Sandia Labs, Albuquerque
1994	Yao Yang	Modeling Group, Hughes Aircraft, Fort Worth
1994	Michael Parks	Symantec Corporation
1996	Jane Wu	Housewife and mother
1996	Ken He	KLA Systems (also supervised by Bob Taylor)
1997	I. S. Dhillon	Comp. Sci. Dept., U. Texas, Austin (also supervised by J. W. Demmel)
2002	E. Barszcz	NASA, Ames, Comp. Sci. Dept., UC, Santa Cruz
2007	Carla Ferreira	Math. Dept., Univ. Minho, Portugal

K. C. Ng (1982)

Argument Reduction for periodic functions.

E.g.:

$$\exp(z) = \exp(z - 2k\pi i), \quad k = 1, 2, \ldots$$

Primary strip: $-\pi < \text{Imag}(z) \leq \pi$

What about matrices?

Given M, consider

$$\left\{ Z : \exp(Z) = \exp(M), \ \text{spectrum}(Z) \text{ in primary strip} \right\}$$

Find Z.

Can define Z using the Jordan Form

$$J_\lambda \longrightarrow J_\lambda - 2k\pi i \mathrm{I} \qquad \text{(suitable } k\text{)}$$

Not practical.

Given U, upper triangular, then $\exists\,!\, S$, upper triangular, satisfying

$$\begin{cases} US = SU \\ -\pi < \mathrm{Imag}(s_{jj}) \leq \pi \end{cases}$$

Cost: $\frac{1}{3}n^3$ *ops* for $n \times n$ S

E.g.:

$$s_{j,j+1} = \begin{cases} u_{j,j+1} & \text{if } u_{jj} = u_{j+1,j+1} \qquad \text{else} \\ u_{j,j+1} \dfrac{s_{jj} - s_{j+1,j+1}}{u_{jj} - u_{j+1,j+1}} & \end{cases}$$

Z. Alex Liu (1991)

Cholesky Factorization for hermitian A:

$$A = L\Omega L^*$$

where L is lower triangular and
Ω is a s.s.p. matrix (signed symmetric permutation)
e.g.:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Extra: if A is tridiagonal then Ω is a direct sum of signed reversal (or flip) matrices

G. W. Stewart: $\not{\mathrm{I}}$

HR and XHR

Every real balanced tridiagonal J may be written as

$$J = \Omega T$$

where $\Omega = \text{diag}(\pm 1)$ and T is symmetric.
J is Ω-symmetric.

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- HR transform (Angelika Bunse-Gerstner)

$$J = HR \longrightarrow \hat{J} = RH = H^{-1}JH$$

H is Ω, $\hat{\Omega}$-orthogonal: $H^t \Omega H = \hat{\Omega}$
R is upper triangular, non-negative diagonal
Ω, $\hat{\Omega} = \mathrm{diag}(\pm 1)$

$$J^t \Omega J = R^t \hat{\Omega} R$$

Properties: $\hat{\Omega}\hat{J}$ is symmetric and tridiagonal.
Transform does not always exist.

- XHR transform (Alex Liu)

Replace diag(± 1) by s.s.p. in definition of Ω.

XHR transform always exists

Need H as a product of plane Ω-rotations, i.e., sinh and cosh as well as sine and cosine.

Implemented it in MAPLE, refused to use FORTRAN!

BONUS 1 For unreduced tridiagonals all s.s.p. Ω's are direct sums of flip matrices

$$\begin{bmatrix}1\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix} \quad \text{etc}$$

BONUS 2 Can generate unreduced balanced tridiagonals with spectrum $\{0\}$

E.g.:

$$\begin{bmatrix} 0 & -1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & -1 & -1 & & \\ & & 1 & 1 & 1 & \\ & & & 1 & 0 & -1 \\ & & & & 1 & 0 \end{bmatrix}$$

Tzon-Tzer Lu (1992)

Thesis accepted for publication by LAA.
I withdrew it in order to provide improved bounds!

Physics: 1D Schrödinger operator with symmetric single-well potential $V(x)$ in interval of length d and Dirichlet boundary conditions

$$\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{d^2}$$

Discrete problem: $n \times n$ tridiagonal, 1's next to diagonal, diagonal entries in $[0, \omega]$, $\omega > 4$

Lu proved:

$$\lambda_{i+1} - \lambda_i \geq 2\frac{(\omega - \omega^{-1})^2}{\omega^n}$$

provided $\omega > 2.88n$

[Conjecture: only need $\omega > n\log(\log n)$]

Needs estimates of decay in eigenvector entries.

BNP conjectured:

extremal diagonal is

$$(\omega, 0, 0, 0, \ldots, 0, 0, 0, \omega)$$

Lu proved:

only true if $\frac{n}{\omega}$ large enough, otherwise extremal diag is

$$(\omega, \omega, 0, 0, \ldots, 0, 0, \omega, \omega)$$

CHAPTER FOUR

ARITHMETIC

Seeing there is nothing (right well beloved Students in the Mathematickes) that is so troublesome to Mathematicall practise, nor that doth more molest and hinder Calculators, then the Multiplications, Divisions, square and cubical Extractions of great numbers, which besides the tedious expence of time are for the most part subject to many slippery errors. I began therefore to consider in my minde, by what certaine and ready Art I might remove those hindrances.

—JOHN NAPIER (1614)

I do hate sums. There is no greater mistake than to call arithmetic an exact science. There are . . . hidden laws of number which it requires a mind like mine to perceive. For instance, if you add a sum from the bottom up, and then again from the top down, the result is always different.

—MRS. LA TOUCHE (19th century)

I cannot conceive that anybody will require multiplications at the rate of 40,000, or even 4,000 per hour; such a revolutionary change as the octonary scale should not be imposed upon mankind in general for the sake of a few individuals.

—F. H. WALES (1936)

Most numerical analysts have no interest in arithmetic.

—B. PARLETT (1979)

THE CHIEF PURPOSE of this chapter is to make a careful study of the four basic processes of arithmetic: addition, subtraction, multiplication, and division. Many people regard arithmetic as a trivial thing that children learn and computers do, but we will see that arithmetic is a fascinating topic with many interesting facets. It is important to make a thorough study of efficient methods for calculating with numbers, since arithmetic underlies so many computer applications.

Arithmetic is, in fact, a lively subject that has played an important part in the history of the world, and it still is undergoing rapid development. In this chapter, we shall analyze algorithms for doing arithmetic operations on many types of quantities, such as "floating point" numbers, extremely large numbers, fractions (rational numbers), polynomials, and power series; and we will also discuss related topics such as radix conversion, factoring of numbers, and the evaluation of polynomials.

Figure: *The Art of Computer Programming, Volume 2*
Donald E. Knuth