GEOMETRIC LITTLEWOOD-RICHARDSON RULES (EQUIVARIANT COHOMOLOGY, K-THEORY, AND FLAG VARIETIES): DEGENERATIONS AND MATROID SHIFTING

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Abstract. Littlewood-Richardson numbers have many interpretations, including as structure coefficients of the cohomology rings of Grassmannians. The geometric Littlewood-Richardson rule is a direct geometric interpretation of these coefficients by degenerations. Allen Knutson and I extend this rule by reinterpretating the degenerations simply in terms of linear equations in Plucker coordinates. Then degenerations correspond to shifting of matroids (a notion from extremal combinatorics), and this turns out to be a fruitful way to package the subtle geometry of the degenerations. This allows us to give an equivariant geometric interpretation of the puzzles of Knutson and Tao, to prove a conjectural rule in K-theory, to conjecture a rule in equivariant K-theory, and hopefully to tackle the general case of the flag variety (which might have interesting interpretations in terms of Coskun's Mondrian tableaux). I will work through an example illustrating the key idea of shifting in action. This is work in progress.

Littlewood-Richardson numbers appear in

- geometry: cohomology ring of Grassmannians
- representation theory
- combinatorics
- symmetric functions

Littlewood-Richardson rule: combinatorial description of Littlewood-Richardson numbers.

Let $G(k,n)$ be the variety parameterizing k-planes in n-space. Cohomology may be generalized to

- Knutson-Tao puzzles
- K-theory
- Buch's set-valued tableaux
- equivariant versions of these

One can also generalize to flag varieties (Coskun).

Write $G(k,n) = \mathbb{G}(k-1,n-1)$, so $\mathbb{G}(k-1,n-1)$ is the variety parameterizing projective $(k-1)$-planes in $\mathbb{P}^{n-1}$.

Fix a *flag* in $\mathbb{C}^n$: a chain of subspaces $F_\cdot = \{F_0 \subset F_1 \subset \cdots \subset F_n\}$ with $\dim F_i = i$.

Consider a k-plane V_k in $\mathbb{C}^n$. What are the possibilities for the list of numbers $\dim V_i \cap F_i$? For example, if $k = 3$ and $n = 5$, then one possibility is

i	0	1	2	3	4	5
$\dim V_i \cap F_i$	0	1	1	2	2	3

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We label this $\{1,3,5\}$ to indicate the jumps. This corresponds to a sequence of up and right moves, where the up moves occur at the same positions as the jumps. This yields the partition $2+1=3$.

Definition 0.1. The *Schubert cell* $\Omega_\alpha(F_\cdot)$ is the subset of $G(k,n)$ such that the partition obtained is α.

We have that $\Omega_\alpha(F_\cdot)$ is an affine space $\mathbb{C}^?$, and $\operatorname{codim}\Omega_\alpha(F_\cdot) = |\alpha|$.

Definition 0.2. The *Schubert variety* is the Zariski closure $\overline{\Omega}_\alpha(F_\cdot)$ of the Schubert cell in $G(k,n)$.

The elements $\left[\overline{\Omega}_\alpha(F_\cdot)\right] \in H_\cdot(G(k,n))$ form a basis for $H_*(G(k,n))$. If α and β have complementary dimension, then

$$\left[\overline{\Omega}_\alpha(F_\cdot)\right] \cup \left[\overline{\Omega}_\beta(F_\cdot)\right] = \delta_{\alpha,\beta^\vee}[\mathrm{pt}].$$

We have

$$\left[\overline{\Omega}_\alpha(F_\cdot)\right] \cup \left[\overline{\Omega}_\beta(F_\cdot)\right] = \sum c^\gamma_{\alpha\beta}\left[\overline{\Omega}_\gamma(F_\cdot)\right]$$

for some nonnegative integers $c^\gamma_{\alpha\beta}$. Equivalently,

$$\begin{aligned} c^\gamma_{\alpha\beta} &= \left[\overline{\Omega}_\alpha(F_\cdot)\right] \cup \left[\overline{\Omega}_\beta(F_\cdot)\right] \cup \left[\overline{\Omega}_{\gamma^\vee}(F_\cdot)\right] \\ &= \#\overline{\Omega}_\alpha(F'_\cdot) \cap \overline{\Omega}_\beta(F''_\cdot) \cap \overline{\Omega}_\gamma(F'''_\cdot) \\ &\geq 0. \end{aligned}$$

Given two flags, one moving (green) and one fixed (red), we will degenerate one in slow motion until it coincides with the other. Here is an example in $\mathbb{C}^4$ (which we draw as $\mathbb{P}^3_{\mathbb{R}}$)). Each flag consists of a point, line, and plane.

Mnemonic: plane; line, plane; point, line, plane.

1. Swivel the green plane around the green line until it contains the red point.
2. Rotate the green line in the green plane until it contains the red point. Swivel the green plane around the green line until it contains the red line.
3. Slide the green point along the green line until it is contained in the red plane. Rotate the green line until it is contained in the red plane. Swivel the green plane around the green line until it coincides with the red plane.

This leads to a calculation in homology that computes the Littlewood-Richardson numbers. We will now

1. define intermediate varieties.
2. describe how they degenerate.
3. start and end appropriately.

Consider the example $\mathbb{G}(1,3) = G(2,4)$, which is of dimension 4. The Plücker embedding gives a closed immersion of this in $\mathbb{P}^5$. Choose a basis $\bar{e}_1, \cdots, \bar{e}_4$ of $\mathbb{C}^4$. Then $\bigwedge^2 \mathbb{C}^4$ has as basis $\bar{e}_i \wedge \bar{e}_j$ for $i<j$. Send a 2-plane to the wedge of any basis: this is well-defined point in $\mathbb{P}^5 = \mathbb{P}\left(\bigwedge^2 \mathbb{C}^4\right)$. The image is defined by the equation

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0.$$

Every Schubert variety is defined in the Grassmannian by variable equations, like $x_{34}=0$, but not conversely: in fact, the subvarieties cut out by variable equations satisfy Murphy's law.

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