CASTELNUOVO-MUMFORD REGULARITY AND LINKAGE

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ABSTRACT. We give bounds on the Castelnuovo-Mumford regularity of varieties in projective space in terms of the degrees of their defining equations. This generalizes work by Bertram, Ein and Lazarsfeld, who had treated the case of smooth varieties in characteristic zero. Instead, we allow mild singularities and have no assumption on the characteristic. Our proofs use linkage, Kodaira vanishing, and a result by Karen Smith in positive characteristic.

Let k be any field. Let $S = k[x_0, \ldots, x_n]$. Let $\mathfrak{m} = (x_0, \ldots, x_n)$.

1. Regularity

Let M be a finite graded S-module. Consider the minimal resolution:

$$\cdots \to \bigoplus S(-j)^{\beta_{ij}} \to \cdots \to \bigoplus S(-j)^{\beta_{0j}} \to M \to 0.$$

Define

$$b_i := \max\{j : \beta_{ij} \neq 0\}$$
$$a_i := \max\{j : H^i_{\mathfrak{m}}(M)_j \neq 0\}$$
$$\operatorname{reg} M := \max\{b_i - i\} = \max\{a_i + i\}.$$

Let I be a homogeneous S-ideal. Let $R = S/I$. Let $X = \operatorname{Proj} R \subset \mathbb{P}^n_k$; its coordinate ring is $R/H^0_{\mathfrak{m}}(R)$.

Bayer-Stillman: If $\operatorname{char} k = 0$, then $\operatorname{reg} I = b_0(\operatorname{gin}(I))$.

2. Bounds for regularity

What might we want to bound regularity in terms of?

- $d = \dim R = \dim X + 1$
- $e = e(R) = \deg X$
- $g = \operatorname{ht} I = \operatorname{codim} X$
- degrees of the homogeneous minimal generators of I; call these $\delta_1 \geq \cdots \geq \delta_s \geq 2$.

If R is Cohen-Macaulay, then $\operatorname{reg} R \leq e - g$.

Conjecture 2.1 (Eisenbud-Goto)**.** If $k = \overline{k}$ and R is a domain, then $\operatorname{reg} R \leq e - g$.

It has been proved roughly up to dimension 7.

Theorem 2.2 (Mumford)**.** *If* $\operatorname{char} k = 0$ *and R is reduced, equidimensional, and has at worst isolated singularities (i.e., X is smooth), then* $\operatorname{reg} R \leq d(e-2)+1$.

If R is Cohen-Macaulay, then $\operatorname{reg} R \leq \delta_1 + \cdots + \delta_g - g$.

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Theorem 2.3 (Bertram-Ein-Lazarsfeld). *If* $\operatorname{char} k = 0$ *and* R *is equidimensional with isolated singularities, then* $\operatorname{reg} R/H^0_{\mathfrak{m}}(R) \le \delta_1 + \cdots + \delta_g - g$.

Galligo, Giusti, Caviglia-Sbarra showed that $\operatorname{reg} R \le (2\delta_1)^{2^{n-1}}$. This is essentially sharp.

3. Results

Assume that $d \ge 2$. Assume that R is not a complete intersection. Assume that $R \not\simeq S/x_0K$ where K is $\mathfrak{m}$ or a complete intersection of 3 forms with $n = 2$. Let I^{unm} be the unmixed part of I. Let $R^{\text{unm}} = S/I^{\text{unm}}$.

Theorem 3.1. *Assume that*

- $\operatorname{char} k = 0$*,*
- R *is equidimensional,*
- $R_{\mathfrak{p}}$ *is a complete intersection if* $\dim R/\mathfrak{p} \ge 2$*, and*
- $R_{\mathfrak{p}}$ *has only rational singularities if* $\dim R/\mathfrak{p} \ge 3$ *(equivalently, the non-rational locus* $\operatorname{NR}(X)$ *of* X *is contained on a curve).*

Then $\operatorname{reg}(R^{\text{unm}}) \le \delta_1 + \cdots + \delta_g - g - 1$.

Theorem 3.2. *Assume that for all primes* $\mathfrak{p} \ne \mathfrak{m}$*,*

- $R_{\mathfrak{p}}$ *is a complete intersection, and*
- $R_{\mathfrak{p}}$ *has at worst rational singularities if* $\operatorname{char} k = 0$*, F-rational singularities if* $\operatorname{char} k > 0$.

Then $\operatorname{reg}(R) \le \frac{(d+1)!}{2}(\delta_1 + \cdots + \delta_g - g - 1)$.

Definition 3.3. Let $(A, \mathfrak{m})$ be a local domain that is essentially of finite type (i.e., a localization of a finitely generated algebra) over a field k. Let $d = \dim A$. Let K be the quotient field of A. Suppose $\operatorname{char} k = p > 0$. Let J be an ideal generated by a system of parameters in A. Let A^+ be the integral closure of A in an algebraic closure of the quotient field. Then A is said to be *F-rational* if and only if $J = (JA^+) \cap A$, or equivalently A is Cohen-Macaulay and $H^d_{\mathfrak{m}}(A)$ is a simple $A[F]$-module.

If instead $\operatorname{char} k = 0$, then A is said to be *of F-rational type* if and only if A is F-rational after reduction to any sufficiently large characteristic p.

Say that A has only *rational singularities* if A is normal, Cohen-Macaulay, and $f_*\omega_Y = \omega_Z$ where $Y \xrightarrow{f} Z = \operatorname{Spec} A$ is a resolution of singularities.

The condition "of F-rational type" was shown by Karen Smith to imply, and shown by Hara and Mehta-Srinivas to be implied by "rational singularities".

Example 3.4. Suppose that $\operatorname{char} k = 0$ and $A = R_{\mathfrak{m}}$ has isolated singularities. Then A has rational singularities if and only if A is Cohen-Macaulay and $\operatorname{reg} R < d$.

4. Proofs

We use linkage. By extending the ground field k, we may assume that k is infinite. Suppose $I \subset S = k[x_0, \ldots, x_n]$, $R = S/I$, $g = \operatorname{ht} I$, and $d = \dim R$. There exists a regular sequence $\alpha_1, \ldots, \alpha_g$ in I of degrees $\delta_1, \ldots, \delta_g$. Let $\mathfrak{a} = (\alpha_1, \ldots, \alpha_g) \subset I$. Define the *link* as $I' = \mathfrak{a} : I$. Let $R' = S/I'$. Note that

- $\mathfrak{a} : I' = I^{\text{unm}}$.

- $\omega_{R'} = \mathrm{Hom}_S(R', \omega_{R/\mathfrak{a}}) = \frac{I^{\mathrm{unm}}}{\mathfrak{a}}(-n-1+\delta_1+\cdots+\delta_g)$.

Proof of Theorem 3.1. We need to show that $\operatorname{reg}\omega_{R'} \le d$. This looks more like a Kodaira vanishing theorem. In fact one needs a better version of it, but such a version is available.

We are assuming $\operatorname{char} k = 0$. For simplicity, suppose that R is a domain. Let $Y \xrightarrow{f} X$ be a resolution of singularities. We have

$$0 \to \bigoplus_i H^0(X, f_*\omega_Y(i)) \to \omega_R \to C \to 0$$

where $\operatorname{Supp}(C) \subset \operatorname{NR}(R)$. Ohsawa (generalized by Kollár) proved

$$\operatorname{reg} \bigoplus_i H^0(X, f_*\omega_Y(i)) \le d.$$

From this one obtains an improved Kodaira vanishing theorem: If $\operatorname{char} k = 0$ and $\dim \operatorname{NR}(\operatorname{Spec} R) \le 1$, then $\operatorname{reg}\omega_R \le d$. To complete the proof, use Theorem 4.1 below. □

Let $\operatorname{NCI}(R)$ be the non-complete-intersection locus of R.

Theorem 4.1. *Assume that*

- $\operatorname{char} k = 0$,
- *R is equidimensional,*
- $\dim \operatorname{NCI}(R) \le t$, *and*
- $\dim \operatorname{NR}(R) \le t+1$.

Then for "generic" $\alpha_1, \dots, \alpha_g$, we have $\dim \operatorname{NR}(R') \le t$.

Proof of Theorem 3.2. Use induction on d by passing to $X \cap X'$ for generic $\alpha_1, \dots, \alpha_g$. We have

$$0 \to \frac{S}{\mathfrak{a}} \to \frac{S}{I} \oplus \frac{S}{I'} \to \frac{S}{I+I'} \to 0,$$

and $\dim S/(I+I') < d$.

The goal is to control the degrees of the generators of $I+I'$ up to finite length.

Recall:

$$\omega_R = \frac{I'}{\mathfrak{a}}(-n-1+\delta_1+\cdots+\delta_g).$$

The goal is now to obtain

$$\bigoplus R(-d) \to \omega_R \to C \to 0$$

with $\dim C \le 0$. Suppose that k is perfect and R is reduced. The map $\bigwedge^d \Omega_{R/k} \to \omega_R$ is neither injective nor surjective in general, but the cokernel C is supported on the singular locus of R. If we could replace the singular locus by the non-F-rational locus $\operatorname{NFR}(R)$, then we would be done, but we cannot do this. Instead we use a weaker Kodaira vanishing theorem in characteristic p due to Karen Smith:

$$\bigoplus R(-d) \to \omega_R \to C \to 0$$

with $\operatorname{Supp}(C) \subseteq \operatorname{NFR}(R)$. In order to keep the induction going, we use Theorem 4.2 below. □

Theorem 4.2. *If* $\dim \operatorname{NCI}(R) \le t$ *and* $\dim \operatorname{NFR}(R) \le t$, *then* $\dim \operatorname{NFR}(S/(I+I')) \le t$.