THE MULTIPLICITY CONJECTURE AND THE STRUCTURE OF BETTI DIAGRAMS

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ABSTRACT. A homogeneous ideal I in a polynomial ring $R = k[x_0, ..., x_n]$ determines a projective scheme and also a homogeneous coordinate ring R/I. Since R/I is graded, we can take the minimal free resolution of R/I, and each slot of this resolution will be a direct sum of twists of the polynomial ring: $\bigoplus R(-a_i)$. The multiplicity conjecture of Herzog, Huneke, and Srinivasan states that if you look only at the largest twist from each slot, you can find an upper bound for the degree of X; a similar formula provides a conjectural lower bound. Despite intense efforts over the past couple decades, progress on the multiplicity conjecture has been slow. In this talk, I'll explain why the multiplicity conjecture has been so elusive, outline the progress that has been made, and look at a recent fresh approach to the problem.

Let $R = k[x_0, \dots, x_n]$. Let $I \subseteq R$ be a homogeneous ideal. Let $c = \operatorname{codim} I$. Let $e(R/I)$ be the multiplicity, i.e., the degree of $\operatorname{Proj}(R/I)$ in $\operatorname{Proj} R = \mathbb{P}^n$.

The minimal free resolution:

$$R/I \leftarrow R \leftarrow \bigoplus_j R(-j)^{\beta_{1,j}} \leftarrow \cdots \bigoplus_j R(-j)^{\beta_{p,j}} \leftarrow 0.$$

Let $\beta(R/I)$ be the matrix

$$\begin{pmatrix} 1 & \beta_{1,1} & \beta_{2,2} & \cdots \\ 0 & \beta_{1,2} & \beta_{2,3} & \cdots \\ 0 & \beta_{1,3} & \beta_{2,4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The number $p = \operatorname{pd}(R/I)$ is called the *projective dimension*.

Definition 0.1. A module M is *Cohen-Macaulay* if $\operatorname{pd}(M) = \operatorname{codim}(\operatorname{ann} M)$.

Here is the multiplicity conjecture:

Conjecture 0.2 (Herzog-Huneke-Srinivasan 1998)**.** Let the resolution of R/I be as above. Let $m_i = \min\{j : \beta_{i,j} \neq 0\}$ and let $M_i = \max\{j : \beta_{i,j} \neq 0\}$. Then

(1) If R/I is Cohen-Macaulay, then

$$\frac{\prod_{i=1}^c m_i}{c!} \leq e(R/I) \leq \frac{\prod_{i=1}^c M_i}{c!}$$

(2) For general R/I, we still have

$$e(R/I) \leq \frac{\prod_{i=1}^c M_i}{c!}.$$

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Example 0.3. Let I be the ideal of 3 points in $\mathbb{P}^3$. Then

$$\beta(R/I) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 2 \end{pmatrix}.$$

We compute $m_1 = 1$, $m_2 = 3$, $m_3 = 4$ and $M_1 = 2$, $M_2 = 3$, $M_3 = 4$. Also $e(R/I) = 3$ and $c = \operatorname{codim}(I) = 3$. The bound gives

$$\frac{1 \cdot 3 \cdot 4}{3!} \leq e(R/I) \leq \frac{2}{3} 43!.$$

Example 0.4. Equidimensionality is important for the lower bound. Suppose $I = I_1 \cap I_2 \subseteq k[x_0, x_1, x_2, x_3]$. Let $I_1 = \langle \ell \rangle$. Let $I_2 = \langle f_1, f_2 \rangle$ where $\deg f_i = N$. We have $V(I) = V(I_1) \cup V(I_2)$. We have $\operatorname{codim}(I) = 1$ and $e(R/I) = 1$, but $m_1 \geq N$.

Observe: The table $\beta(R/I)$ determines the Hilbert polynomial, and hence $\operatorname{codim}(I)$ and $e(R/I)$. It also determines the m_i and M_i and $\operatorname{pd}(R/I)$.

The conjecture is hard because Betti diagrams are mysterious.

What's known? Known cases:

(1) Complete intersections: the resolution is the Koszul complex.
(2) Cohen-Macaulay codimension 2: use the Hilbert-Burch theorem. The Hilbert-Burch theorem gives

$$R/I \leftarrow R \leftarrow \bigoplus_{j=1}^{N+1} R(-a_j) \leftarrow \bigoplus_{j=1}^{N} R(-b_j)$$

with $m_1 \geq N$.
(3) Gorenstein codimension 3: use the Buchsbaum-Eisenbud theorem.
(4) Some monomial ideals.
(5) Pure resolutions (Betti diagram has one entry in each column).

Ideals of smooth curves, monomial ideals are not known cases, in general.

Theorem 0.5 (Peskine-Szpiro 1974)**.** *Suppose that M has minimal resolution*

$$0 \leftarrow M \leftarrow \bigoplus_j R(-j)^{\beta_{0,j}} \leftarrow \cdots \leftarrow \bigoplus_j R(-j)^{\beta_{p,j}} \leftarrow 0.$$

Let $c = \operatorname{codim}(M)$. Then we have the following linear relations of the Betti numbers:

(0): $\sum_{i,j} (-1)^i \beta_{ij} = 0$
(1): $\sum_{i,j} (-1)^i j \beta_{ij} = 0$
(1) $\cdots$
$(c-1)$: $\sum_{i,j} (-1)^i j^{c-1} \beta_{ij} = 0$

Proof. Let $F_m(t) = \sum_{i,j} (-1)^i \beta_{i,j} t^j$. The Hilbert series is $\operatorname{HS}_m(t) = F_m(t)/(1-t)^{n+1} = Q(t)/(1-t)^{n+1-c}$ for some polynomial $Q(t)$. So $(1-t)^c$ divides $F_m(t)$. Thus $\frac{d^k}{dt^k} F_M(t)\Big|_{t=1} = 0$ for $k = 0, \cdots, c-1$. $\square$

Definition 0.6. A *diagram* D is a $\mathbb{Q}$-matrix. A diagram is *pure* if there is exactly one nonzero entry in each column. A diagram is *normalized* if the first column sums to 1. If the first column of D sums to a nonzero number, the normalization $\overline{D}$ is D divided by this sum, so $\overline{D}$ is normalized.

Example 0.7. The diagram
$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$
is pure of type $d = (0, 1, 3, 4)$.

The diagrams $\beta(M)$ and $\beta(R/I)$ are normalized.

Fix c and fix strictly increasing sequences of integers $\underline{d} = (\underline{d}_0, \ldots, \underline{d}_c)$ and $\overline{d} = (\overline{d}_0, \ldots, \overline{d}_c)$. Let $B_{\underline{d},\overline{d}}$ be the set of $\beta(M)$ where M is Cohen-Macaulay of codimension c, and $\beta_{i,j}(M) \neq 0$ implies $0 \leq i \leq c$ and $\underline{d}_i \leq j \leq \overline{d}_i$. Let $\overline{B}_{\underline{d},\overline{d}}$ be the set of $\overline{\beta}(M)$ where $\beta(M)$ is in $B_{\underline{d},\overline{d}}$.

Proposition 0.8. *The set $\overline{B}_{\underline{d},\overline{d}}$ is convex.*

Proof. The set $B_{\underline{d},\overline{d}}$ is closed under addition: $\beta(M_1 \oplus M_2) = \beta(M_1) + \beta(M_2)$. □

Let $\Pi_{\underline{d},\overline{d}}$ be the set of normalized pure diagrams of type $\delta = (\delta_0, \ldots, \delta_c)$ where $\underline{d}_i \leq \delta_i \leq \overline{d}_i$ satisfies the Peskine-Szpiro relations.

Conjecture 0.9 (Boij-Söiderberg 2007)**.** The set $\overline{B}_{\underline{d},\overline{d}}$ is the convex hull of $\Pi_{\underline{d},\overline{d}}$.

Example 0.10. Let $M' = k[x, y]/(x^2, xy, y^4)$. Then
$$\beta(M) = \begin{pmatrix} 1 & & \\ & 2 & 1 \\ & & \\ & 1 & 1 \end{pmatrix}$$
and $\underline{d} = (0, 2, 3)$ and $\overline{d} = (0, 4, 5)$. We have
$$\beta(M) = \frac{1}{2}\begin{pmatrix} 1 & & \\ & 3 & 2 \\ & & \\ & & \end{pmatrix} + \frac{3}{10}\begin{pmatrix} 1 & & \\ & 5/3 & \\ & & \\ & & 2/3 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 1 & & \\ & & \\ & & \\ & 5 & 4 \end{pmatrix}.$$
The three matrices on the right correspond to $k[x, y]/(x, y)^2$, $(x^4, x^2y^2, y^4)/(x^6, x^3y^3, y^6)$, and $k[x, y]/(x, y)^4$.