

HILBERT'S 14TH PROBLEM AND COX RINGS

ANA-MARIA CASTRAVET (WITH J. TEVELEV)

Hilbert's 14th problem: Let G be an algebraic group acting linearly on a polynomial algebra $S = \mathbb{C}[x_1, \dots, x_n]$. Is S^G finitely generated?

Hilbert: Yes, if G is reductive.

Weitzenböck: Yes, if $G = \mathbb{G}_a$.

Nagata: Counterexample with $G = \mathbb{G}_a^{13}$.

Mukai: Counterexamples with $G = \mathbb{G}_a^g$ for $g \geq 3$.

Define the *Nagata action* as follows: $(t_1, \dots, t_r) \in \mathbb{C}^r$ acts on $\mathbb{C}[x_1, \dots, x_r, y_1, \dots, y_r]$ by $x_i \mapsto x_i$ and $y_i \mapsto y_i + t_i x_i$. Let $G \subseteq \mathbb{C}^r$ be a linear subspace of dimension g . Mukai showed that if G is general and $r - 3 \geq g \geq 3$, then S^G is not finitely generated.

Mukai: How about \mathbb{G}_a^2 ?

Nagata: The coordinate vectors in \mathbb{C}^r give points p_1, \dots, p_r in $\mathbb{P}(\mathbb{C}^r/G) = \mathbb{P}^{r-g-1}$. Let X be the blow-up $\text{Bl}_{p_1, \dots, p_r} \mathbb{P}^{r-g-1}$. Then there is an isomorphism $S^G \simeq \text{Cox}(X)$.

Let X be a smooth projective variety such that $\text{Pic } X$ is a free abelian group, generated by D_1, \dots, D_r . Define

$$\begin{aligned} \text{Cox}(X) &= \bigoplus_{(m_1, \dots, m_r) \in \mathbb{Z}^r} H^0(m_1 D_1 + \dots + m_r D_r) \\ \text{Eff}(X) &= \text{support of } \text{Cox}(X). \end{aligned}$$

This was known before Cox as the total coordinate ring, but it was Cox who proved that if X is toric, then $\text{Cox}(X)$ is a polynomial ring. Keel and Hu proved conversely that if $\text{Cox}(X)$ is a polynomial ring, then X is toric. This is part of a more general theory by Keel-Hu: $\text{Cox}(X)$ is finitely generated if and only if X is a ‘‘Mori dream space’’ (and in this case X is a GIT quotient -by the action of a torus- of the affine variety with coordinate ring $\text{Cox}(X)$). Call D moving if its base locus is of codimension ≥ 2 . Let $\text{Mov}(D)$ be the cone of moving divisors. Then $\text{Mov}(D) = \bigcup_{X' \sim X} \text{Nef}(X')$ (here $X' \sim X$ means isomorphic in codimension 1). Call $\text{Nef}(X')$ *Mori chambers*.

Birkar-Cascini-Hacon-McKernan: If X is *log-Fano* (that is, $-K - \Delta$ is ample and (X, Δ) is Kawamata log-terminal), then X is a Mori dream space.

Example 0.1. Let $X = \text{Bl}_r(\mathbb{P}^2)$, the blow-up of \mathbb{P}^2 at $r \geq 8$ general points. If D is a curve in X , define $\text{deg}(D) = D \cdot (-K)$. If D is a (-1) -curve (or if $r = 8$ and $D = -K$), then adjunction shows that $\text{deg } D = 1$. The converse holds. Batyrev-Popov showed that $\text{Cox}(X)$ is generated in degree 1 and conjectured the relations are quadratic. Stillman and his collaborators proved this in some cases.

Example 0.2. Let $X = \text{Bl}_{n+2}(\mathbb{P}^n)$. Then $\text{Cox}(X) = \bigoplus H^0(\mathbb{G}, \mathcal{O}_{\mathbb{G}}(m))$, where $\mathbb{G} = G(2, n+3)$ is the Grassmannian, and $\mathcal{O}_{\mathbb{G}}(1)$ is defined for the Plücker embedding into $\mathbb{P}^{\binom{n+3}{2}-1}$. Then $\text{Cox}(X) \simeq S^G$ where $G = \mathbb{G}_a \subseteq \mathbb{C}^{n+2}$. We have $S^G = \mathbb{C}[x_1, \dots, x_{n+2}, x_i y_j : i \neq j]$.

Date: April 24, 2007.

Mukai: Let $X = \text{Bl}_r(\mathbb{P}^n)$ be general. Let $G \subset \mathbb{C}^r$ be $(r-n-1)$ -dimensional. Then $\text{Cox}(X)$ is not finitely generated if

$$\frac{1}{2} + \frac{1}{n+1} + \frac{1}{r-n-1} \leq 1.$$

For $n = 2$, this applies for $r \geq 9$. For $n \geq 5$, this applies for $r \geq n + 4$.

Remaining case: $r = n + 3$.

Theorem 0.3. *Let $C \subseteq \mathbb{P}^n$ be a rational normal curve of degree n . Let p_1, \dots, p_r be distinct points on C , and let $X = \text{Bl}_{p_1, \dots, p_r} \mathbb{P}^n$. Then $\text{Pic}(X)$ is a free abelian group with basis H, E_1, \dots, E_r and $\text{Cox}(X)$ is generated by sections corresponding to E_i and $kH - k \sum_{i \in I} E_i - (k-1) \sum_{i \in I^c} E_i$ for all $1 \leq k \leq n/2 + 1$ and $I \subseteq \{1, \dots, r\}$ with $|I| = n + 2 - 2k$.*

What are the divisors?

Suppose $X = \text{Bl}_r \mathbb{P}^n$. Then $\text{Pic}(X)$ has \mathbb{Z} -basis H, E_1, \dots, E_r , and $-K_X = (n+1)H - (n-1) \sum E_i$. Define a bilinear form $(\ , \)$ on $\text{Pic} X$ as follows: $(H, H) = n-1$ and $(H, E_i) = 0$ and $(E_i, E_j) = -\delta_{ij}$. This agrees with the intersection pairing when $n = 2$.

In $(-K)^\perp$ this defines a root system with simple roots

$$\alpha_1 = E_1 - E_2, \alpha_2 = E_2 - E_3, \dots, \alpha_{r-1} = E_{r-1} - E_r, \alpha_r = H - \sum_{i=1}^{n+1} E_i.$$

This gives a three-legged Dynkin diagram $T_{2, n+1, r-n-1}$ with α_r connected to a chain $\alpha_1, \dots, \alpha_{r-1}$ at α_{n+1} . Let W be the Weyl group. Fact: W is finite if and only if

$$\frac{1}{2} + \frac{1}{n+1} + \frac{1}{r-n-1} > 1.$$

Claims:

- (1) $W.E_1$ consists of effective divisors.
- (2) If p_1, \dots, p_r are general, then $W.E_1$ are (-1) -divisors, i.e., exceptional divisors on $X' \sim X$.
- (3) (-1) -divisors must appear in any set of generators of $\text{Eff}(X)$.

Theorem 0.4. *Let $X_{a,b,c} := \text{Bl}_{b+c}(\mathbb{P}^{c-1})^{a-1}$. The following are equivalent:*

- (1) $\text{Cox}(X_{a,b,c})$ is finitely generated.
- (2) The Weyl group $T_{a,b,c}$ is finite.

We have $\deg D = (D, -K) \cdot \frac{1}{\dim X - 1}$. If D is in $W.E_1$, then $\deg D = 1$.

Suppose that $\text{Cox}(X_{a,b-1,c})$ is generated in degree 1 and $T_{a,b,c}$ is finite. We want to prove that $\text{Cox}(X_{a,b,c})$ is finitely generated. Let $R := \text{Cox}(X_{a,b,c}) \supset R' := \mathbb{C}[x_E | E \in W.E_1]$. Then $R' \subset R \subset \bigcap_E R'_{X_E} = \bigoplus_{m \geq 0} H^0(\text{Proj } R', \mathcal{O}(m))$.

View $X_{a,b,c}$ as a blowup of $X_{a,b-1,c}$ at a point, and let E be the exceptional divisor in $X_{a,b,c}$. Let $R_0 = \text{Cox}(X_{a,b-1,c})$. Then $R[X_E^{-1}] \simeq R_0[T, T^{-1}]$. Then $R \subset R'_{X_E}$. We get $R_{(x_E)} \simeq R_0$.