GENERIC PROJECTIONS OF PROJECTIVE VARIETIES

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1. General projections

Let X be a smooth projective variety of dimension n. Suppose $X \subseteq \mathbb{P}^r$. Let $\pi\colon X \to Y \subseteq \mathbb{P}^{n+1}$ be a general projection from a general $(r-n-2)$-plane in $\mathbb{P}^r$. Then π is birational.

Question 1.1. What can be said about fibers of π?

We may assume that $r = 2n+1$.

If $n = 1$, and $X \subseteq \mathbb{P}^3$ is non-degenerate, then we are discussing $\pi\colon X \to Y \subseteq \mathbb{P}^2$. Here X has a 2-dimensional family of 2-secant lines and at most a 1-dimensional family of 3-secant lines. And Y has nodes.

If $n = 2$, then $X \subseteq \mathbb{P}^5$, and ℓ is a general line, and $\pi\colon X \to Y \subseteq \mathbb{P}^3$ is projection from ℓ, then there exists $C_\ell \subset X$ (double curve of π) such that $\pi\colon X \setminus C_\ell \to X$ is generically 2-to-1, and 3-to-1 at finitely many points, and nowhere 4-to-1.

Theorem 1.2 (Roberts)**.** *If X is a smooth n-dimensional variety, then there exists an embedding $i\colon X \hookrightarrow \mathbb{P}^r$ such that for a general projection $\pi\colon i(X) \to Y \subset \mathbb{P}^{n+1}$, all curvilinear fibers have length $\leq n+1$. (Call $S \subseteq \mathbb{P}^r$* curvilinear *if S is contained in a smooth curve.)*

Theorem 1.3 (Mather)**.** *Suppose that $X^n \subseteq \mathbb{P}^r$ and $\pi\colon X \to Y \subseteq \mathbb{P}^{n+1}$. If $n \leq 14$, then all fibers have length $\leq n+1$.*

Remark 1.4. One cannot find a linear bound on the length of a fiber. Suppose we have $\pi\colon X \to Y \subseteq \mathbb{P}^{n+1}$ as usual. We have the derivative $d\pi\colon T_X \to \pi^* T_{\mathbb{P}^{n+1}}$. Let $A_x = d\pi \otimes \mathbb{C}(x)$. If $\operatorname{rank} A_x \leq n-k$, then $\hat{\mathcal{O}}_{\pi^{-1}(y),X} = \frac{\mathbb{C}[[x_1,\dots,x_k]]}{(f_1,\dots,f_{k+1})}$ where the f_i are of order ≥ 2, and the length is $\geq 2^k - 2^{k-2} = \frac{3}{4}2^k$. Let $X_k = \{x \in X : \operatorname{rank} A_x \leq n-k\}$. Then the expected codimension of X_k is $k(k+1)$.

Lazarsfeld's non-emptiness theorem states that if $k(k+1) \leq n$ and $T_X^* \otimes \pi^* T_{\mathbb{P}^{n+1}}$ is ample then $X_k \neq \emptyset$. If k is about $\sqrt{n} - 1$, then $\ell \geq \frac{3}{4} 2^{\sqrt{n}-1}$.

Conjecture 1.5 (Eisenbud)**.** Suppose $X^n \subseteq \mathbb{P}^r$ and $\pi\colon X \to Y \subseteq \mathbb{P}^{n+1}$ is the general projection, then all the fibers of π have regularity $\leq n+1$.

If S is a 0-dimensional subscheme of $\mathbb{P}^r$, then $\operatorname{reg}(S)$ is the minimum k such that $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k+1)) \to H^0(S, \mathcal{O}_S(k+1))$ is surjective. So $\operatorname{reg}(S) \leq \operatorname{length}(S)$. If S is contained in a line then $\operatorname{reg}(S) = \operatorname{length}(S)$.

Theorem 1.6. *If $X^n \subseteq \mathbb{P}^r$ and $\pi\colon X^n \to Y^n \subseteq \mathbb{P}^{n+1}$ is a general projection, and $Y_i := \{y \in Y : \pi^{-1}(y)$ contains a local complete intersection of length $\geq i\}$, then $\dim \overline{Y}_i \leq n+1-i$. In particular, there is no l.c.i. fiber of length $\geq n+2$.*

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Curvilinear implies l.c.i. Either $Y_i = \emptyset$ or $\dim \overline{Y}_i = n + 1 - i$.

Suppose $X^n \subseteq \mathbb{P}^r$. Let Z_ℓ be the subvariety of $\mathbb{P}^r$ swept out by ℓ-secant lines of X.

Corollary 1.7. *$\dim Z_{n+2} \leq n + 1$.*

Proof. Suppose not, so $\dim Z_{n+2} \geq n + 2$. Then for any $(r - n - 2)$-plane Λ, there exists $P \in \Lambda \cap Z_{n+2}$. ☐

Theorem 1.8. *$\dim Z_\ell \leq n\ell/(\ell - 1) + 1$.*

This bound is sharp: If $t \leq n\ell/(\ell - 1) + 1$ (so $\ell(t - n - 1) \leq t - 1$), then look at general hypersurfaces $X_1, \ldots, X_{t-n}$ of degree ℓ in $\mathbb{P}^t$, and let $X = X_1 \cap \cdots \cap X_{t-n}$, so X is smooth of dimension n. Then

$$\sum_{1 \leq i \leq t-n-1} \deg X_i = \ell(t - n - 1) \leq t - 1,$$

and this inequality implies that the intersection is covered by lines. Any such line is an ℓ-secant line.

We can do the same with any hypersurfaces $Y_1, \ldots, Y_{t-n-1}$ in the linear system generated by $X_1, \ldots, X_{t-n}$. So these lines cover $\mathbb{P}^t$.

Proof of Theorems 1.6 and 1.8. Let Hilb_k be the Hilbert scheme of k-planes in $\mathbb{P}^r$. Let $\mathcal{Y}_{k,\ell} \subset \mathrm{Hilb}_k$ be the subscheme parametrizing k-planes λ such that $\mathrm{length}(\Lambda \cap X) \geq \ell$. Suppose $[\Lambda] \in \mathcal{Y}_{k,\ell}$. Then $T_{\mathrm{Hilb}_k,[\Lambda]} \simeq H^0(\Lambda, N_{\Lambda/\mathbb{P}^r})$. There is an exact sequence of sheaves

$$0 \to F \to N_{\Lambda/\mathbb{P}^r} \to Q \to 0$$

such that

$$T_{\mathcal{Y}_{k,\ell},[\Lambda]} \simeq H^0(\Lambda, F).$$

and Q is supported at $\Lambda \cap X$.. and if $\Lambda \cap X$ is a l.c.i., then $h^0(\Lambda, Q) \geq (r - n - k)\ell$. If $k = 1$ and $L = \Lambda$, then

$$0 \to F \to \mathcal{O}_L^{r-1}(1) \to Q \to 0.$$

Write $F = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{r-1})$. Now $a_i \geq -\ell + 1$. If $h^0(Q) \geq (r - n - 1)\ell$, then $\sum a_i \leq r - 1 - (r - n - 1)\ell$, so the number of nonnegative a_i is $\leq n\ell/(\ell - 1)$. ☐

2. Applications

2.1. **Regularity of subvarieties of $\mathbb{P}^n$.** Let X be a projective variety in $\mathbb{P}^r$. Then X is k-regular if $H^i(\mathbb{P}^r, \mathcal{I}_X(k - i)) = 0$ for all $i \geq 1$. One can show that if X is k-regular, then X is $(k + 1)$-regular. Define $\mathrm{reg}(X)$ to be the smallest k such that X is k-regular.

Conjecture 2.1 (Eisenbud, Goto 1984)**.** *If X is nondegenerate, then $\mathrm{reg}(X) \leq d - e + 1$, where $d := \deg X$ and $e := r - n = \mathrm{codim}\, X$.*

This, if true, is sharp.

What is known about this conjecture?

Bertram, Ein, Lazarsfeld: If X is smooth, then

$$\mathrm{reg}(X) \leq \min\{e, n + 1\}(d - 1) + 1.$$

If $f \colon M \to \mathbb{P}^r$ is the blow up of $\mathbb{P}^r$ along X, then the Kawamata-Viehweg vanishing theorem implies $H^i(\mathbb{P}^r, \mathcal{I}_X(m)) = H^i(M, f^*\mathcal{O}(m)(-E)) = 0$, where E is the exceptional divisor.

For curves (possibly singular), then G-L-P proved the conjecture.
For smooth surfaces, the conjecture holds (Lazarsfeld).
If $n = 3$, then $\operatorname{reg}(X) \leq (d - e + 1) + 1$.
If $n = 4$, then $\operatorname{reg}(X) \leq (d - e + 1) + 4$.
If $n = 5$, then $\operatorname{reg}(X) \leq (d - e + 1) + 10$.
If $n = 6$, then $\operatorname{reg}(X) \leq (d - e + 1) + 20$.

Proposition 2.2. *If Conjecture 1.5 holds, then for any r, n there is a constant $c_{n,r}$ such that* $\operatorname{reg}(X) \leq (d - e + 1) + c_{n,r}$.

Explicitly, if $\dim V = r - n - 1$, then one can take

$$c_{n,r} = \sum_{j=3}^{n+1} (j - 2) \dim \operatorname{Sym}^i(V).$$

2.2. **Regularity of powers of ideals.** Let $S = \mathbb{C}[x_0, \dots, x_n]$. Let $P = (x_0, \dots, x_n)$. Let $I \subseteq S$ be a homogeneous ideal. What is $\operatorname{reg}(I^t)$?

Theorem 2.3 (C.,H.,T.,K.). $\operatorname{reg}(I^t) = at + b$ *for sufficiently large t.*

Assume that I is generated by forms of degree d. Let $I = (F_0, \dots, F_m)$. Then $a = d$. What can we say about b?

Theorem 2.4 (Eisenbud, Harris, Huneke). *Assume moreover that there exists k such that $P^k \subseteq I$. If $\phi \colon \mathbb{P}^n \to \mathbb{P}^m$ is given by $(F_0 : \cdots : F_m)$, then* $\max_{y \in \mathbb{P}^m} \operatorname{reg} \phi^{-1}(y) - 1 = b$.

What is this value of b when $F_0, \dots, F_m$ are *general* forms?
Let $v_{n,d} \colon \mathbb{P}^n \to \mathbb{P}^r$ be the Veronese embedding, so $r = \binom{n+d}{d} - 1$. Let $X_{n,d} = v_{n,d}(\mathbb{P}^n)$. Let $\pi_{n,d} \colon X_{n,d} \to Y_{n,d} \subseteq \mathbb{P}^m$ be a general projection. Then the general ϕ is $\pi_{n,d} \circ v_{n,d}$. If $m \geq 2n + 1$, then $b = 0$. If $n + 1 \leq m \leq 2n$, then we do not know anything. If $m = n + 1$, Conjecture 1.5 implies $b \leq d(n + 1) - 1$, but maybe the true answer is independent of d.