HILBERT SCHEME OF POINTS

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Abstract. $\mathrm{Hilb}_n(X)$ parametrizes 0-dimensional, degree n subschemes of X. Even for X equal to affine or projective space over an algebraically closed field, $\mathrm{Hilb}_n(X)$ can be quite complicated. Much research has been done on the case of smooth surfaces, for which $\mathrm{Hilb}_n(X)$ is well-behaved, and in particular smooth. However, while it is known that higher dimensional cases are more complicated, even basic questions are open, such as the irreducibility of $\mathrm{Hilb}_n(\mathbb{A}^3)$ for many values of n. I will introduce the Hilbert schemes of points, and then present examples illustrating the complex behavior that is possible.

We will talk about old work of Anthony Iarrobino, and new work of a Berkeley group: Dustin Cartwright, Bianca Viray, Dan Erman, Jonah Blasiak, Mauricio Velasco (graduate student at Cornell), Bjorn Poonen, Bernd Sturmfels, David Eisenbud.

Assume that k is an algebraically closed field of characteristic 0.

Let $\mathrm{Hilb}_n(\mathbb{P}^d)$ be the scheme whose T-valued points are flat families over T in $\mathbb{P}^d$ with 0-dimensional degree-n fibers.

Define $\mathrm{Hilb}_n(\mathbb{A}^d)$ to be the open subscheme of $\mathrm{Hilb}_n(\mathbb{P}^d)$ parametrizing subschemes contained in $\mathbb{A}^d$. The closed points of $\mathrm{Hilb}_n(\mathbb{A}^d)$ correspond to ideals in $k[x_1, \dots, x_n] =: A$ of colength n.

Question 0.1. When is $\mathrm{Hilb}_n(\mathbb{A}^d) =: H_n^d$ irreducible?

When d is 1 or 2, the scheme H_n^d is smooth and irreducible. If $d \geq 3$ and $n \gg 0$, then H_n^d is reducible: one sees this by considering ideals generated by k homogeneous forms of degree r and all homogeneous forms of degree $r+1$. For example, H_{96}^3 is reducible and H_{21}^4 is reducible. Let U_n^d be the open subscheme of H_n^d corresponding to radical ideals. Let $R_n^d = \overline{U_n^d}$ be the Zariski closure of U_n^d.

Example 0.2. Let $I = \langle x^2, xy, y^2, z^2, zw, w^2, xz + yw \rangle$. This gives a point of H_8^4. But it is not in R_8^4 (Iarrobino-Emsalem 1972): using the formula $T_I(H_n^d) = \mathrm{Hom}_A(I, A/I)$, we find that $\dim_k T_I(H_8^4) = 25 < 32 = \dim R_8^4$. (This has been extended to all characteristics.) This point belongs to a component birational to $\mathrm{Gr}(7, 10) \times \mathbb{A}^4$. We have a closed immersion $\phi\colon \mathrm{Gr}(7, 10) \times \mathbb{A}^4 \to H_8^4$?

What does $R_8^4 \cap Z$ look like? It will be the product of some subscheme of $\mathrm{Gr}(7, 10)$ and $\mathbb{A}^4$.

Conjecture 0.3 (Sturmfels)**.** View $\mathrm{Gr}(7, 10)$ (2-forms in $k[x, y, z, w]$) as $\mathrm{Gr}(3, \mathrm{Hom}(S_2 k[x, y, z, w], k))$, where S_2 denotes the degree-2 part of the polynomial ring. An element on the right corresponds to a triple (A, B, C) of 4×4 symmetric matrices. Then $\phi^{-1}(R_8^4)$ is defined by the

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Pfaffian of

$$\begin{pmatrix} 0 & A & B \\ -A & 0 & C \\ -B & -C & 0 \end{pmatrix}.$$

Let $\mathfrak{m} = \langle x_1, \ldots, x_d \rangle \subset k[x_1, \ldots, x_d]$. Is there an ideal of the form $\langle k \text{ } r\text{-forms} \rangle + \mathfrak{m}^{r+1}$ not in R_n^3 for $n < 96$?

No (Viray): We have

$$\{\{n \text{ distinct points in } \mathbb{A}^3\}\} \to U_n^3 \to H_n^3,$$

where the last map takes the $(1,1,1)$-initial ideal. We use Haiman coordinates. Let D be a n-element subset of $\mathbb{N}^d$. Write $D = \{\alpha_1, \ldots, \alpha_n\}$, where each α_i is in $\mathbb{N}^d$. We identify each α_i with a monomial with the corresponding exponents. Define

$$\Delta_D \colon \{\{n \text{ distinct points in } \mathbb{A}^3\}\} \to k$$
$$\left(\vec{b}_1, \ldots, \vec{b}_n\right) \mapsto \det\left(\vec{b}_j^{\alpha_i}\right).$$

(Strictly speaking, we should choose an ordering of the α_i; otherwise Δ_D is determined only up to sign.) This is a generalization of a Vandermonde determinant.

Let $\lambda = \{1, x, y, z, \ldots, x^a y^b z^c\}$, where $a + b + c = r$. Let λ' be λ with one degree r monomial replaced by a different one, α' (in the same position). Then $\Delta_{\lambda'}/\Delta_\lambda$ is well-defined, independent of the ordering of the monomials.

Let I be the ideal of the union of the n points $\vec{b}_1, \ldots, \vec{b}_n$ in $\mathbb{A}^3$. We have $\Delta_\lambda \neq 0$ if and only if the set of monomials λ span the k-vector space A/I, where The ratio $\Delta_{\lambda'}/\Delta_\lambda$ equals the $\vec{x}^{\alpha_n}$-coordinate of the image in A/I of $\vec{x}^{\alpha'_n}$, with respect to the basis of A/I given by λ.

If we know $\frac{\Delta_{\lambda'}}{\Delta_\lambda}(\vec{b}_1, \ldots, \vec{b}_n)$ for all degree r monomials α'_r, then we know the degree r part of an element $\vec{x}^{\alpha'_n} - \cdots$ in I; thus we know an element of the initial ideal $\text{in}_{(1,1,1)}(I)$. If we do this for all degree r monomials in λ with all possible degree r monomials outside λ, then we know what $\text{in}_{(1,1,1)}(I)$ is. These $\Delta_{\lambda'}/\Delta_\lambda$ give coordinates onto an affine patch V of the Grassmannian, where V parametrizes linear subspaces arising as graphs of functions from the k-span of the degree-r monomials in λ to the k-span of the degree-r monomials outside λ.

Look at the Jacobian of this map from the open subset $U_n^{3'}$ of U_n^3 where $\Delta_\lambda \neq 0$ to the open subset V of the Grassmannian. If the Jacobian has rank equal to $\dim V$ at some $(\vec{b}_1, \ldots, \vec{b}_r)$, then ϕ is smooth at $(\vec{b}_1, \ldots, \vec{b}_r)$ and hence dominant.

For $n = 1, \ldots, 95$, we compute the rank of this Jacobian at some random $(\vec{b}_1, \ldots, \vec{b}_r)$. The rank is always $\dim V$.

Why does this happen in 3-space, but not in 4-space?

Iarrobino proved also that H_3^8 is irreducible. Mazzola (1980) proved that H_n^d is irreducible for $n \leq 7$ and any d.

We know by another argument that H_n^3 is reducible for $n \geq 78$.