# THE PAST, PRESENT, AND FUTURE OF LIMIT LINEAR SERIES

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Abstract. Many classical questions on curves can be either rephrased in terms of or approached via linear series, which are objects closely related to morphisms to projective spaces. Many of the most fundamental results on linear series of the past 30 years have been proved via degeneration techniques, and there is plenty of evidence that such techniques will continue to play an important role.

We will give a survey of the most powerful and general such degeneration technique, the theory of limit linear series originally developed by Eisenbud and Harris in the 1980s. After describing the Eisenbud-Harris point of view, we will sketch a newer construction which functorializes and compactifies the Eisenbud-Harris construction, discuss ongoing work (with Deepak Khosla) to construct a universal limit linear series moduli space, and suggest further directions for generalization which are likely to prove important in the future.

## 1. Past

**Definition 1.1.** A *linear series* of degree $d$ and dimension $r$ (also known as a $g^r_d$) on a (smooth projective) curve $C$ over a field $k$ is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle on $C$ of degree $d$, and $V \subseteq \Gamma(C, \mathcal{L})$ is a subspace with $\dim_k V = r + 1$.

For $P \in C$ we have the *vanishing sequence* $0 \leq a_0(P) < \cdots < a_r(P) \leq d$ defined by looking at orders of vanishing at $P$ of sections in $V$.

**Examples 1.2.**

- If $a_0(P) = 0$ for all $P$, then we say that $(\mathcal{L}, V)$ is *basepoint-free*, and we then get a map $C \to \mathbb{P}^r_k$ of degree $d$, well-defined up to an element of $\mathrm{Aut}(\mathbb{P}^r_k)$, and conversely.
- If $r = 1$, and $(\mathcal{L}, V)$ is basepoint-free, then $a_1(P)$ is the ramification index at $P$ of the corresponding morphism $f : C \to \mathbb{P}^1_k$.

Applications of linear series theory:

- Brill-Noether: For which $(g, r, d)$ does every curve of genus $g$ have a map of degree $\leq d$ to $\mathbb{P}^r$? (Castelnuovo, Severi, Kempf, Kleiman, Laksov, Griffiths-Harris)
- What kind of Weierstrass points can a curve have? (Eisenbud-Harris) (Weierstrass points are points for which the vanishing sequence for the canonical sheaf differs from the sequence at the generic point.)
- Is $\mathcal{M}_g$ unirational? Is $\mathcal{M}_g$ of general type? Yes for $g \geq 24$ (Harris-Mumford, Eisenbud-Harris, ...)

All of these have been approached via linear series, and more specifically via degeneration techniques.

The most general powerful degeneration technique is the theory of limit linear series of Eisenbud and Harris.

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Assume that we have a family of curves $X \to B$ with one singular fiber $X_0$, and that the total space $X$ is regular. Suppose that we have $(\mathcal{L}, V)$ on $X - X_0$. Then $\mathcal{L}$ extends to $\overline{\mathcal{L}}$ on $X$. We can take $\overline{V}$ to be the limit of $V$ in $H^0(X, \overline{\mathcal{L}})$. This is not completely well-defined, because we could twist $\mathcal{L}$ by $\mathcal{O}(Y)$ where $Y$ is a component of $X_0$. In other words, the limit $(\overline{\mathcal{L}}_{X_0}, \overline{V}_{X_0})$ is not uniquely defined.

If $X_0$ consists of two components $Y$ and $Z$ intersecting transversely at a point, then twisting by $\mathcal{O}(Y)$ raises the degree of $\overline{\mathcal{L}}_{X_0}$ on $Z$ by 1, so it lowers the degree on $Y$ by 1. Fact: twisting by $Y$ is the only ambiguity, so for all $i \in \mathbb{Z}$, there exists a unique $\overline{\mathcal{L}}$ with degree $i$ on $Y$ and degree $d - i$ on $Z$.

The Eisenbud-Harris solution: Consider limits that have degree $d$ on $Y$ and degree 0 on $Z$, and vice versa. Then if we restrict to $Y$, there is no loss of information: we get a $g^r_d$ on $Y$.

So given a family of $g^r_d$'s on $X - X_0$, we get a pair of $g^r_d$'s, one on $Y$ and one on $Z$. Given such a pair, when does it arise as a limit? The answer, roughly, is as follows: Let $a^Y_i$ be the vanishing sequence at the node of the $g^r_d$ on $Y$, and define $a^Z_i$ similarly. Then $a^Y_i + a^Z_{r-i} \geq d$ for all $i \in [0, r]$. A pair of $g^r_d$'s satisfying this inequality is a *limit linear series.* If $a^Y_i + a^Z_{r-i} = d$ for all $i \in [0, r]$, then we say that the limit linear series is *refined.*

**Theorem 1.3** (Eisenbud-Harris)**.** *Given a reasonable family $X \to B$ (in particular, of compact type), we have a $B$-scheme $G^{r,EH}_d(X/B)$ that parametrizes $g^r_d$'s on smooth fibers and refined limit series on reducible fibers. If $G^{r,EH}_d(X_0/k)$ has the expected dimension given by Brill-Noether, then the refined limit series can be smoothed to nearby fibers.*

There is an analogue of this theorem for the case in which we impose ramification (or prescribe vanishing sequences) along sections. This means that the Eisenbud-Harris theory is inductive.

But it is not functorial, and not a proper space. We can often work around the lack of properness by blowing up $X$.

## 2. New construction

Suppose $X/B$ is as before, with $X_0$ consisting of two components $Y$ and $Z$ intersecting transversely at a point (or something slightly more general). We will define a proper scheme $G^r_d(X/B)$ representing a certain functor (compatible with base change).

Then keep track of all extensions of $\mathcal{L}$ with nonnegative degree on $Y$ and $Z$. Explicitly, we consider a collection of pairs $(\mathcal{L}^i, V^i)$ on $X_0$ having degree $i$ on $Y$, indexed by $i \in [0, d]$ such that $V^i$ maps to $V^{i+1}$ under $\mathcal{L}^i \to \mathcal{L}^{i+1} = \mathcal{L}^i(Y)$. and such that $V^{i+1}$ maps to $V^i$ under $\mathcal{L}^{i+1} \to \mathcal{L}^i$ given by $\mathcal{L}^i \simeq \mathcal{L}^{i+1}(Z)$.

**Theorem 2.1.** *This has the same properties as $G^{r,EH}_d$. (The hard part is smoothing.)*

Problem: Not inductive.

**Theorem 2.2.** *On $X_0$, $G^{r,EH}_d \simeq G^r_d$ on the refined locus. Furthermore, there is a surjective set-theoretic map $G^r_d \to G^{r,EH}_d$ (including crude series on the right), and there is a bound on the fiber dimensions strong enough to give (for instance) an inductive proof of the Brill-Noether theorem.*

**Theorem 2.3** (Helm, Osserman)**.** *If $G^r_d$ has the expected dimension, then it is Cohen-Macaulay, and hence flat if every fiber has the expected dimension.*

I am currently working with Deepak Khosla to generalize to all curves of compact type, and to construct a universal $G^r_d$ stack. This is useful for computing divisors on $\mathcal{M}_g$ and constructing counterexamples to the Harris-Morrison slope conjecture.

Future work: generalize to higher-rank vector bundles and higher-dimensional varieties.

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