p-ADIC ÉTALE SHEAVES ON CHARACTERISTIC-p CURVES

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ABSTRACT. I will briefly introduce the class of étale sheaves. This class has a number of important applications in characteristic-p geometry, including a proof of the Weil conjectures.

There is a lot of literature that is concerned with ℓ-adic étale sheaves on characteristic-p schemes, where ℓ is a prime different from p. The behavior of p-adic étale sheaves is fundamentally different from the behavior of ℓ-adic étale sheaves. I will discuss a theorem about ℓ-adic étale sheaves on characteristic-p curves, and then I will present a partial extension that applies to p-adic étale sheaves.

Let k be an algebraically closed field of characteristic $p > 0$. Let X be a smooth projective genus-g curve over k. Let R be a finite commutative ring. Let $\mathrm{Et}^c(X, R)$ be the category of constructible étale sheaves of R-modules on X.

There is a similar category $\mathrm{Zar}(X, R)$ of Zariski sheaves.

The hypothesis "Suppose $p \nmid \#R$" is common in the literature. But $H^1_{\mathrm{et}}(X; \mathbb{F}_p)$ also is interesting, because its order matches that of the kernel of $J(k) \xrightarrow{p} J(k)$.

Example of an étale sheaf: Consider the squaring map $f\colon \mathbb{P}^1 \to \mathbb{P}^1$. Then $f_*\mathbb{F}_p$ is an étale sheaf that does not come from a Zariski sheaf. We have $\dim(f_*\mathbb{F}_p)_x = 2$ for all points x except the branch points 0 and ∞.

1. Euler characteristics in $\mathrm{Et}(X, R)$

Theorem 1.1 (Grothendieck-Ogg-Shafarevich). *Let ℓ be a prime different from p. Let $\mathcal{G} \in \mathrm{Ob}\,\mathrm{Et}(X; \mathbb{F}_\ell)$ be a sheaf whose sections all have open support. (The hypothesis eliminates skyscraper sheaves, which are easy to deal with anyway.) Let n be the generic rank of $\mathcal{G}$. Then*

$$\chi(X, \mathcal{G}) = n(2 - 2g) - \sum_{x \in |X|} \mathcal{S}(\mathcal{G}_{(x)}).$$

Notation:

- $\chi(X, \mathcal{G}) = h^0(X, \mathcal{G}) - h^1(X, \mathcal{G}) + h^2(X, \mathcal{G})$. (The nontrivial term is $h^1(X, \mathcal{G})$.)
- $\mathcal{G}_{(x)}$ is the pullback of $\mathcal{G}$ via a morphism $i\colon \operatorname{Spec} k[[t]] \to X$ whose image includes x.
- $\mathcal{S}(\mathcal{G}_{(x)}) := n - \dim \mathcal{G}_x + \mathrm{Swan}(\mathcal{G}_{(x)})$

Example 1.2. Let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the squaring map and $\mathcal{G} = f_*\mathbb{F}_\ell$. Then $n = 2$, and $\dim \mathcal{G}_x = 2$ except at the two ramified points, where it is 1. If $p > 2$, then $\mathrm{Swan}(\mathcal{G}_{(x)}) = 0$, so $\mathcal{S}(\mathcal{G}_{(x)}) = 0$ except at the two ramified points, where it is 1. Thus

$$\chi(\mathbb{P}^1, \mathcal{G}) = 2(2) - (1 + 1) = 2.$$

We have $h^0(\mathbb{P}^1, \mathcal{G}) = h^0(\mathbb{P}^1, \mathbb{F}_\ell) = 1$, $h^1(\mathbb{P}^1, \mathcal{G}) = h^1(\mathbb{P}^1, \mathbb{F}_\ell) = 0$, and $h^2(\mathbb{P}^1, \mathcal{G}) = h^1(\mathbb{P}^1, \mathbb{F}_\ell) = 0$, because a spectral sequence degenerates.

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Let $E \to \operatorname{Spec} k$ be an elliptic curve. Then $\chi(E, \mathbb{F}_p)$ can be 0 or 1.

Proposal (R. Pink): find a lower bound for Euler characteristic of $\mathbb{F}_p$-étale sheaves on X in the form of the Grothendieck-Ogg-Shafarevich formula.

Tool: The Riemann-Hilbert correspondence of Emerton and Kisin: there are maps

$$\operatorname{Et}(X, \mathbb{F}_p) \xrightarrow{M} \{\text{unit } \mathcal{O}_{F,X}\text{-modules}\}$$

$$\{\text{unit } \mathcal{O}_{F,X}\text{-modules}\} \xrightarrow{\text{sol}} \operatorname{Et}(X, \mathbb{F}_p).$$

Theorem 1.3. *Let $\mathcal{G} \in \operatorname{Ob} \operatorname{Et}(X, \mathbb{F}_p)$ be a rank-n sheaf whose sections all have open support. Then*

$$\chi(X, \mathcal{G}) \geq n(1-g) - \sum_{x \in |X|} \mathcal{C}(M(\mathcal{G}_{(x)})).$$

The proof uses the Riemann-Roch theorem.

Example 1.4. Let $f \colon \mathbb{P}^1 \to \mathbb{P}^1$ be an Artin-Schreier cover. Let $\mathcal{G} = f_* \mathbb{F}_p$. Then

$$\chi(\mathbb{P}^1, \mathcal{G}) = 1 - 0 + 0 = 1.$$

On the right hand side of the theorem, we get

$$p(1-0) - \mathcal{C}(M(\mathcal{G}_{(\infty)})) = p - (p-1) = 1.$$

2. The minimal root index

Let $A = k[[t]]$ and $K = k((t))$. There is a local Riemann-Hilbert correspondence

$$M \colon \operatorname{Et}(\operatorname{Spec} A, \mathbb{F}_p) \to \{\text{unit (left) } A[F]\text{-modules}\}.$$

Here $A[F]$ is a twisted polynomial ring in which $Fa = a^p F$ for all $a \in A$. The definition is $M(\mathcal{G}_{(x)}) := \operatorname{Hom}(\mathcal{G}_{(x)}, \mathcal{O}_{\operatorname{Spec} A, \text{et}})$.

Definition 2.1. An A-submodule M_0 of M is called a *root* if

1. M_0 generates M as a left $A[F]$-module.
2. The A-submodule M_1 of M generated by $F(M_0)$ contains M_0.

Definition 2.2. The *index* of the minimal root M_0 of M is

$$\frac{\dim_k(M_1/M_0)}{p-1}.$$

(It is not always an integer.)

The minimal root of K is $A[t^{-1}]$, and its index is 1.

Definition 2.3. If M is a unit $A[F]$-module, then $\mathcal{C}(M)$ is the index of the minimal root of M.

3. Riemann-Hilbert correspondence

Let N be a complex manifold. Then local systems of finite-dimensional $\mathbb{C}$-vector spaces on N correspond to finite-dimensional bundles with integrable connection on N. (To go left, take the sheaf of horizontal sections.)