PROJECTIONS AND THE REGULARITY OF POWERS

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Abstract. Every variety is birational, by a generic projection, to a hypersurface. When we make such a projection for a curve or surface, we don't do much damage to the geometry of the variety: the fibers of the projection have small length. But it turns out that as the dimension grows, the fibers may grow exponentially, and not much is known about them. A new algebraic approach bounds the Castelnuovo-Mumford regularity of the fibers in terms of an algebraic invariant of the homomogeneous coordinate ring.

Theorem 0.1 (Cutkosky-Herzog-Trung, Kodiyalam, Trung-Wang)**.** $I \subset S = \mathbb{C}[x_0, \dots, x_n]$ *Let* M *be a finitely generated (in degree* 0*) graded module. Then* $\operatorname{reg}(I^m M) = dm + e$ *for* $m \gg 0$ *and some non-negative integers* d, e*.*

Obvious questions: What is d? Good formulas are known (Kodiyalam, Trung-Wang, Cutkosky-Lazarsfeld-Ein.) What is e? Nothing is known beyond a rather special case (see below) and some examples. How big does m have to be for the equality to hold? Roemer has proven a result in a special case, Ulrich and I have a stronger one in a more special case. *Much* remains to be done.

It turns out that the constant e has something to do with **fibers of projections:**

If X is an r-dimensional smooth projective variety, it can be embedded in $\mathbb{P}^{2r+1}$. A generic projection to $\mathbb{P}^{r+1}$ will give a hypersurface.

Generic projection curvilinear for $r \geq 5$ (rank of $T_X \to \pi^* T_{\mathbb{P}^{r+1}}$ drops by at most one at each point).

Wild hope: for the generic projection, the length of $\pi^{-1}(y)$ is $\leq r+1$ for all y. This is true for $r \leq 5$.

Lazarsfeld: This is false! In general, we should expect the existence of fibers $\pi^{-1}(y)$ of length at least on the order of constant times $2^{\sqrt{r}}$.

Reason: Suppose k is even (the odd case is similar). At a corank k point (that is, a point where the rank of $T_X \to \pi^* T_{\mathbb{P}^{r+1}}$ is at most $r-k$, we have

$$\hat{\mathcal{O}}_{\pi^{-1}\pi(x),x} \cong \mathbb{C}[[z_1, \dots, z_k]]/(k+1 \text{ power series of order} \geq 2).$$

A little calculation shows that the length of this ring is $\geq \binom{k}{k/2} + \binom{k}{k/2+1}$, which grows exponentially with k. Also, the set of corank k points is either empty or of codimension less than or equal to $k(k-1) \sim k^2$, and by theorems of Lazarsfeld and Fulton, the set of these points will be non-empty when when $k(k-1) \geq r$ and the embedding line bundle $\mathcal{O}_X(1)$ is sufficiently positive.

Example 0.2. There exists a smooth projective surface $X^2 \subseteq \mathbb{P}^5$ such that for a generic line L, the projection $\pi_L \colon X \to \mathbb{P}^3$ from L has a fiber with 3 collinear points.

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Let A be a 4×4 symmetric matrix of linear forms. The zero locus of the 3×3 minors is a smooth surface X. The intersection $L \cap (\det M)$ is 4 points. Let p be one of these points. Without loss of generality, the first column of A vanishes identically at p, and it vanishes also on a line L' such that $L' \cap X$ is 3 points (remaining 3×3 minor). These 3 points are in the fiber corresponding to the intersection of the plane spanned by L and L' with X.

Example 0.3 (Beheshti)**.** If $X_1^{n-1} \subseteq \mathbb{P}^n$ is a hypersurface of degree $\leq n-1$ and $p \in X_1$, then there exists a line $p \in L \subseteq X_1$. Let $X^{n-2} = X_1 \cap X_2$ where X_1 and X_2 are both hypersurfaces of degree $d \leq n-1$ in $\mathbb{P}^n$. Let $\pi \colon X \to \mathbb{P}^{n-1}$ be the projection from a generic point p.

Claim: There exists $y \in \mathbb{P}^{n-1}$ such that $\pi^{-1}y$ has $n-1$ points on a line.

Proof: Without loss of generality, $p \in L \subset X_1$. Then $L \cap X = L \cap X_2$, which is $n-1$ points. The regularity of $n-1$ points in a line is $n-1$, which is $\dim X + 1$.

The following two conjectures are equivalent:

Conjecture 0.4. (geometric version) Suppose we have a smooth projective variety $X^r \subseteq \mathbb{P}^N$. Let $\pi \colon X^r \to \mathbb{P}^{r+1}$ be a generic projection. Then $\operatorname{reg} \pi^{-1}(y) \leq r+1$ for all points y.

Conjecture 0.5. (algebraic version) Let S_X be a graded algebra with isolated singularity, and let P_X be its homogeneous maximal ideal. Suppose $\dim S_X = r+1$. Let $\ell_1, \ldots, \ell_{r+2}$ be general linear forms. Let $I = (\ell_1, \ldots, \ell_{r+2})$. Then for $m \gg 0$

$$I^m \supset P_X^{m+r}$$

The equivalence is a consequence of the following result, which is a special case of a theorem about base-point-free linear series:

Theorem 0.6 (Eisenbud, Harris, Huneke)**.** *Let S_X be the homogeneous coordinate ring of a scheme $X \subset \mathbb{P}^n$, and suppose that V is a vector space of degree d forms defining a scheme that is disjoint from X. If $\phi \colon X \to \mathbb{P}(V)$ is the map defined by ϕ, then for $m \gg 0$*

$$\operatorname{reg}((V)^m S_X) = dm + e,$$

where $e = \max_y \operatorname{reg} \phi^{-1}(y) - 1$.