LIMITS OF SUBVARIETIES: SUBSCHEMES VS. BRANCHVARIETIES

ALLEN KNUTSON (JOINT WITH VALERY ALEXEEV)

Let $F \to \mathbb{C}^*$ be a family over $\{t \neq 0\}$. We want to define $\overline{F} \to \mathbb{C}$ and $\overline{F}_0 \subseteq \overline{F}$.

Obvious answer: $\overline{F}_0 = \emptyset$, $\overline{F} = F$. We can rule this out by insisting that $\overline{F} \to \mathbb{C}$ be proper.

Another condition: $\overline{F}_0$ should not contain a component of $\overline{F}$. We can impose this by insisting that $\overline{F} \to \mathbb{C}$ be flat.

Still not unique:

$$\mathbb{C}[a] \times \mathbb{C}[b] \leftarrow \mathbb{C}[t]$$
$$(a, b) \leftarrow t$$

and

$$\mathbb{C}[x, t]/(x^2 - t^2) \leftarrow \mathbb{C}[t].$$

The first has reduced $\overline{F}_0$; the second does not.

Grothendieck says: Embed F as a closed subvariety of $\mathbb{C}^\times \times \mathbb{CP}^n$. Then define $\overline{F}$ as the closure in $\mathbb{C} \times \mathbb{CP}^n$. Bad: Even if the general F_t is reduced, $\overline{F}_0$ might not be.

Definition 0.1. A *branch*variety X of Y is a (geometrically) reduced scheme X with a finite map to Y.

We don't want to give up reducedness of fibers, but we are willing to replace subvariety by branchvariety.

For $\mathbb{C}[x, t]/(x^2 - t)$, the monodromy switching $\sqrt{t}$ and $-\sqrt{t}$ holds the two branches together. Introduce t' where $t'^2 = t$. Pull back to define $\sqrt{F}$:

$$\begin{tikzcd}
\sqrt{\overline{F}} \arrow[r] \arrow[d] & \overline{F} \arrow[d] \\
\mathbb{C}_{t'} \arrow[r, "2:1"] & \mathbb{C}_t
\end{tikzcd}$$

Theorem 0.2. *Let $F \to \operatorname{Spec} \mathbb{C}((t)) \times \mathbb{CP}^n$ be finite and reduced. Then for any sufficiently divisible k, the zero fiber of the normalization of $\widetilde{\sqrt[k]{F}}$ of the family $\overline{\sqrt[k]{F}}$ in the open set $\sqrt[k]{F}$ has reduced fiber $\left(\sqrt[k]{F}\right)_0$, and gives the same answer (for $\left(\sqrt[k]{F}\right)_0$) for all such k.*

Proof. (We are working in characteristic 0.) 0-dimensional case: colliding points. Let $\pi \in S_{\deg f}$ be the monodromy around 0. Let k be the LCM of the cycle lengths in π, so k is the order of π. Then $\sqrt[k]{F}$ is the trivial covering space.

Higher-dimensional case. We want to ensure $\widetilde{\sqrt[k]{F}}_0$ is reduced: R0 (smooth in codimension 0) and S1 (no embedded primes). If $\widetilde{\sqrt[k]{F}}$ were normal, then it would be R1 and S2, so $\widetilde{\sqrt[k]{F}}_0$ is S1. Exercise: True in this case too.

Date: May 2, 2006.

R0 can be checked at a generic point. Reduce to the 0-dimensional case by using a general hyperplane section. So if $k = \prod_{\text{components}}(\deg C)!$, these general points will be smooth. $\square$

Example 0.3. Consider a family of plane conics: $\operatorname{Proj}\mathbb{C}[x,y,z,t]/(y^2 - txz)$ where t (the parameter) has degree 0, over $\operatorname{Spec}\mathbb{C}[t]$. Introduce $t'^2 = t$, and set $y' = y/t'$. We get $\mathbb{C}[x,y',z]/(y'^2 - xz)$.

Example 0.4. 0-dimensional branchvarieties with $\{n \text{ points}\} \to \mathbb{CP}^3$ are the same as collections of points in $\mathbb{CP}^3$ with multiplicities summing to n.

Theorem 0.5. *If F is a flat proper family of branchvarieties whose general fiber has k connected components, then any fiber has k connected components.*

Example 0.6. Two skew lines in $\mathbb{P}^3$ coming together have a non-reduced flat limit in the Hilbert scheme, but the branchvariety limit consists of a union of two disjoint lines mapping non-injectively into $\mathbb{P}^3$.

Definition 0.7. The *(rooted labeled) forest* of a branchvariety X of $\mathbb{CP}^n$. The roots correspond to connected components. Above each vertex, one has one vertex for each connected component of a generic hyperplane section.

Label each vertex by the constant term in the Hilbert polynomial: these are enough to reconstruct the Hilbert polynomial of the branchvariety corresponding to each vertex. For an irreducible variety, the forest is a "palm tree" with one vertex at each level except at the top, where the number of vertices is the degree of the variety.

Theorem 0.8. *Suppose* $\operatorname{char} 0$. *For fixed $\mathbb{CP}^n$ and fixed rooted labeled forest, there is a proper separated moduli stack of such branchvarieties; whose coarse moduli space is an algebraic space (reference for algebraic spaces: Knutson 1971).*

In $\operatorname{char} p$, *instead of the forest, we fix $b_i := \sum_{\text{components } C \text{ of dimension } i} \deg(C)$.*

UNIVERSITY OF CALIFORNIA AT SAN DIEGO