

DESINGULARIZING THE MAIN COMPONENT OF THE SPACE OF GENUS 1 STABLE MAPS TO PROJECTIVE SPACE

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We work over \mathbb{C} . Moduli space of stable maps $\overline{\mathcal{M}}_{g,k}(X, \beta)$ where g refers to the genus of the curves, k is the number of marked points, X is a complex projective manifold, and β is a homology class in $H^2(X, \mathbb{Z})$.

There is an inclusion

$$\overline{\mathcal{M}}_{1,k}(\mathbb{P}^n, d) \hookrightarrow \overline{\mathcal{M}}_{1,k}(\mathbb{P}^n, d)$$

Let $\tilde{M} \rightarrow \overline{\mathcal{M}}_{1,k}(\mathbb{P}^n, d)$ be the desingularization.

- The boundary is a s.n.c. divisor, explicit normal bundle.
- The points of the boundary have geometric meaning.
- It is an isomorphism over $\mathcal{M}_{1,k}(\mathbb{P}^n, d)$.
- Blow up most singular locus, then the second most singular (twist), etc.
- Equivariant for $\text{Aut } \mathbb{P}^n$.
- well-behaved with respect to $\mathbb{P}^m \hookrightarrow \mathbb{P}^n$ and marked points.
- The universal curve $\mathcal{C} \xrightarrow{\rho} \overline{\mathcal{M}}_{1,k}(\mathbb{P}^r, d)$ admits a map $\mathcal{C} \xrightarrow{\pi} \mathbb{P}^n$, and the desingularization transforms the bundle $\rho_*\pi^*\mathcal{O}(a)$ in a nice way.
- Enumerative geometry of curves, localization.
- Relate to Gromov-Witten invariants (also for complete intersections X : work of J. Li and Zinger 2005)
- Algebraic version of “reduced” Gromov-Witten invariants (Zinger 2005)
- May give an approach to proving the physicist’s conjecture on genus 1 Gromov-Witten invariants (Zinger, in progress)

Assume $k = 0$ for simplicity. A *stable map* from a nodal projective curve C to X is a morphism $C \rightarrow X$ such that any contracted genus-0 curve has at least 3 node-branches (points on the normalization of the component that map to a node of C), and $C \rightarrow X$ is not the map from a genus-1 curve to a point. (This means that the morphism has finite automorphism group)

There is a moduli space $\overline{\mathcal{M}}_g(X, \beta)$ of such objects (i.e., a Deligne-Mumford stack). It is proper. Let $\mathcal{M}_g(X, \beta)$ be the substack classifying stable maps with C smooth.

$\overline{\mathcal{M}}_0(\mathbb{P}^n, d)$ is beautiful: irreducible, smooth, and the boundary has simple normal crossings.

$\overline{\mathcal{M}}_g(\mathbb{P}^n, d)$ horrible: arbitrarily singular (Murphy’s Law: see my talk here in 2005).

Where is $\overline{\mathcal{M}}_1(\mathbb{P}^n, d)$ smooth? The open part $\mathcal{M}_1(\mathbb{P}^n, d)$ turns out to be smooth. Check smoothness at $\pi: C \rightarrow \mathbb{P}^n$. Assume that $C \hookrightarrow \mathbb{P}^n$ is a closed immersion of a smooth curve C . The first order deformations are given by $H^0(C, N)$ where N is the normal bundle, and the obstructions are given by $H^1(C, N)$. To calculate these, use the exact sequence

$$0 \rightarrow T_C \rightarrow \pi^*T_{\mathbb{P}^n} \rightarrow N \rightarrow 0.$$

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We get

$$\begin{aligned}
& 0H^0(T_C) \\
& \rightarrow H^0(T_{\mathbb{P}^n}) \rightarrow H^0(N) \rightarrow H^1(T_C) \\
& \rightarrow H^1(T_{\mathbb{P}^n}) \rightarrow H^1(N) \rightarrow 0. \\
& 0 \rightarrow \text{Aut } \pi \rightarrow \text{Aut}(\pi: C \rightarrow \mathbb{P}^n) \rightarrow \text{Aut } C \\
& \rightarrow \text{Def } \pi \rightarrow \text{Def}(\pi: C \rightarrow \mathbb{P}^n) \rightarrow \text{Def } C \\
& \rightarrow \text{ob } \pi \rightarrow \text{ob}(\pi: C \rightarrow \mathbb{P}^n) \rightarrow \text{ob } C
\end{aligned}$$

Smooth if $H^1(C, \pi^*T_{\mathbb{P}^n}) = 0$. Using the Euler sequence, we get

$$0 \rightarrow \pi^*\mathcal{O} \rightarrow \pi^*\mathcal{O}(1)^{\oplus(n+1)} \rightarrow \pi^*T_{\mathbb{P}^n} \rightarrow 0$$

So if $H^1(C, \pi^*\mathcal{O}(1)) = 0$, then smooth. If C has no contracted genus 1 component, smooth.

Example 0.1. $\overline{\mathcal{M}}_1(\mathbb{P}^2, 3)$. The space $\mathcal{M}_1(\mathbb{P}^2, 3)$ is of dimension 9. But $\overline{\mathcal{M}}_1(\mathbb{P}^2, 3)$ also has points corresponding to a genus-0 curve of degree 3 crossing a genus-1 curve of degree 0: these form a 10-dimensional space. Another family: a genus-1 curve of degree 0 with two genus-0 curves of degrees 1 and 2 attached: these form a 9-dimensional space; in fact it corresponds to a component distinct from the others. Another family: a genus-1 curve of degree 0 with three genus-0 curves of degrees 1, 1, 1 attached: these form a 8-dimensional space, but this one is in the closure of $\mathcal{M}_1(\mathbb{P}^2, 3)$. In the intersection of the first two components one has points corresponding a nodal cubic with a genus-1 curve attached at the node. In the intersection of the last two components one has a conic tangent to a line, with a genus-1 curve attached at the point of tangency.

Proposition 0.2 (Vakil 1997). *A map $C \rightarrow \mathbb{P}^1$ with C of genus 1 is smoothable if and only if either*

- (1) *there is no contracted genus-1 component*
- (2) *there is a contracted genus-1 component with attached genus-0 curves having dependent tangent vectors in \mathbb{P}^n .*

In general, one can define substacks such as $\overline{\mathcal{M}}_1(\mathbb{P}^n, d)_{4+6}$ parameterizing a genus-1 curve with genus-0 curves of degrees 4 and 6 attached.

We blow up components in order. Because of the identity $\text{Bl}_X(X \cup Y) = \text{Bl}_{X \cap Y} Y$, some components get blown out of existence, while their intersection with other components get blown up on what remains.

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