A NEW APPROACH TO MORI THEORY

JAMES MCKERNAN

We work over $\mathbb{C}$.

1. Curves

Curves come in three types.

Curve	Geometry	Topology	Differential geometry	Arithmetic
Riemann sphere $\mathbb{P}^1$	$\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2)$	$\pi_1(\mathbb{P}^1) = e$	Ricci curvature $+1$	potentially dense rational points
Elliptic curve E	$\mathrm{Aut}(E)$ is finite ext. of itself	$\pi_1(E) \simeq \mathbb{Z} \oplus \mathbb{Z}$	zero Ricci curvature	potentially dense rational points
$g \geq 2$ C	$\mathrm{Aut}(C)$ finite	$\pi_1(C)$ not almost abelian	Ricci curvature -1	rational points finite

E.g., $x^2 + y^2 = -1$ over $\mathbb{Q}$ has no rational points, but rational points are dense over $\mathbb{Q}(i)$. Potentially dense means that rational points over some finite extension are dense.

Fix $g \geq 2$. There is a nice moduli space $\mathcal{M}_g \subseteq \overline{\mathcal{M}}_g$.

Idea: one single geometric invariant controls the picture: the canonical divisor K_X. By definition, K_X is zeros minus poles of a meromorphic differential: $K_X = c_1(\bigwedge^n TX^*)$.

2. Surfaces

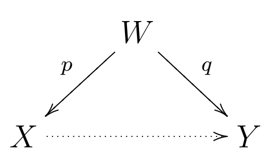
Consider the *Cremona transformation*

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$
$$[X : Y : Z] \mapsto [X^{-1} : Y^{-1} : Z^{-1}] = [YZ : XZ : XY].$$

Let Γ be its graph in $\mathbb{P}^2 \times \mathbb{P}^2$. Then Γ has a hexagon of lines such that projection to either $\mathbb{P}^2$ blows down three of the lines. We have that $-K_{\mathbb{P}^2}$ is ample, and $-K_\Gamma$ is ample. If we blow up ≥ 10 lines, we get a surface S such that K_S does not have a well-defined sign: there exist curves C on S with $K_S.C$ positive or negative.

The aim of the minimal model program (MMP) is to start with Γ or S and replace it by either $\mathbb{P}^2$ or…

Definition 2.1. Let $\phi\colon X \dashrightarrow Y$ be a birational map between two normal projective varieties. Say that ϕ is a *contraction mapping* if ϕ^{-1} does not contract any divisors. Choose a diagram

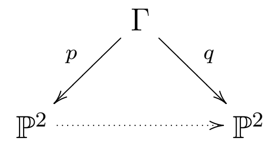


in which W is another normal projective variety, and p and q are birational morphisms. We say that a contraction mapping is *D-negative*, for a Cartier $\mathbb{Q}$-divisor D on X, if $D' := \phi^* D$ is $\mathbb{Q}$-Cartier and if we write $p^*D = q^*D' + E$ then $E \geq 0$. (This is independent of the choice of W.) In this case, we have

$$H^0(X, \mathcal{O}(mD)) \simeq H^0(Y, \mathcal{O}(mD')).$$

Date: April 18, 2006.

In



p and q are contraction mappings, but ϕ is not one.

Conjecture 2.2 (Minimal model conjecture)**.** Let X be a smooth projective variety. Then there is a K_X-negative contraction $\phi\colon X \dashrightarrow Y$ and a morphism $\pi\colon Y \to Z$ (Y and Z need not be smooth) such that either

(1) $K_Y = \pi^* H$ for some ample H on Z
(2) $-K_Y$ is ample on the fibers of π ($\dim Z < \dim Y$).

Examples 2.3. In these, we take $\phi = \mathrm{id}$.

(1) If $X = \mathbb{P}^2$, take $\pi\colon \mathbb{P}^2 \to \mathrm{pt}$.
(2) If $X = \mathbb{P}^1 \times C$ with $g(C) \geq 2$, take $\pi\colon \mathbb{P}^1 \times C \to C$.
(3) If $X = E \times C$ with E elliptic and $g(C) \geq 2$, take $\pi\colon E \times C \to C$; then $K_Y = \pi^* K_C$.
(4) If $X = C \times D$ with $g(C), g(D) \geq 2$, then take $\pi\colon C \times D \to C \times D$ to be the identity.

3. Minimal model program (first version)

Let $X = S$, a surface.

(1) Replace S be a desingularization.
(2) Is K_S nef? I.e., is $K_S.C \geq 0$ for all $C \subseteq S$? If yes, then STOP.
(3) If not, then $\exists \pi\colon S \to T$ such that $-K_S$ is relatively ample and has connected fibers. There are two cases:
    (a) If $\dim T < \dim S$, then STOP.
    (b) If $\dim T = \dim S$, then π is birational; replace S by T and go back to (2).

Remark 3.1. If K_Y is nef, there is a separate conjecture that states that $K_Y = \pi^* H$ with H ample as in the minimal model conjecture.

Why does this process terminate? Because every time we contract a -1 curve, the first betti number goes down by one.

4. 3-folds

Consider the Cremona transformation for 3-folds:

$$\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$$
$$[X : Y : Z : T] \mapsto [X^{-1} : Y^{-1} : Z^{-1} : T^{-1}].$$

It is indeterminate at least at the four points $[1 : 0 : 0 : 0]$ and so on. Blow up $\mathbb{P}^3$ at the four points $\tilde{\mathbb{P}}^3$. The plane $T = 0$ lifts up to the surface with the hexagon. Blow up the other $\mathbb{P}^3$ at its four points. We get a new map $\tilde{\mathbb{P}}^3 \to \tilde{\mathbb{P}}^3$. It is an isomorphism in codimension 1. But certain $\mathbb{P}^1$'s in the indeterminacy locus of this new map get “flopped”.

The group $\mathrm{Bir}(\mathbb{P}^2)$ of birational transformations is generated by $\mathrm{PGL}(3)$ and a single Cremona transformation.

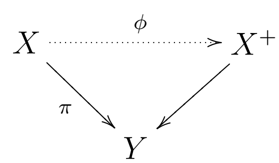
What is $\mathrm{Bir}(\mathbb{P}^3)$?

5. Minimal model program (second version)

Let X be a variety of any dimension.

(1) Desingularize X.
(2) If K_X is nef, then STOP.
(3) Otherwise there exists a contraction $\pi\colon X \to Y$ with $-K_X$ relatively ample. Three cases:
    (a) If $\dim Y < \dim X$, then STOP.
    (b) If $\dim Y = \dim X$, then π is birational.
        (i) The exceptional locus is a divisor. Replace X by Y; go back to (2)
        (ii) π is small; Y is too singular (K_Y is not $\mathbb{Q}$-Cartier). In this case, use a flip $X \dashrightarrow X^+$. Replace X by X^+; go back to (2).

Flip:



with $-K_X$ π-ample small, $-K_{X^+}$ π-ample small.

Conjecture 5.1. Flips exist.

Conjecture 5.2. Flips terminate.

Conjecture 5.3. If K_X is nef, then π is semiample: there exists $\pi\colon X \to Z$ with $K_X = \pi^*H$, H ample.

Conjecture 5.4. Let X be a smooth projective variety. The ring $\bigoplus_{m\in\mathbb{N}} H^0(X, \mathcal{O}_X(mK_X))$ is finitely generated.

Theorem 5.5 (Hacon, McKernan)**.** *(roughly) Flips exist in dimension n, if they terminate in dimension $n-1$.*

If U is quasi-projective, we can embed $U \hookrightarrow X$ such that $D = X - U$ has normal crossings. Consider D with all multiplicities 1, and take $K_X + D$. In the case $\dim U = 1$, and D consisting of d points, then $\deg(K_C + D) = 2g - 2 + d$.

Curve	Geometry	Topology	Arithmetic
$U = \mathbb{A}^1$	$\mathrm{Aut}(\mathbb{A}^1) = \{az + b\}$	$\pi_1(\mathbb{A}^1) = e$	dense integral points
$U = \mathbb{P}^1 - \{0, 1\}$	$\mathrm{Aut}(U)$ finite ext. of itself	$\pi_1(U) \simeq \mathbb{Z}$	potentially dense integral points
$U = \mathbb{P}^1 - \{0, 1, \infty\}$	$\mathrm{Aut}(U) = S_3$	$\pi_1(U) = F_{a,b}$	integral points

Similar statement in last row for $E - \{0\}$.

$\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$. We have $M_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\} \hookrightarrow \mathbb{P}^1 = \overline{M}_{0,4}$. The points in $\overline{M}_{0,4} - M_{0,4}$ classify pairs of $\mathbb{P}^1$'s crossing at a point, with two marked points on each one.

Consider $K_X + \Delta$, where $\Delta = \sum a_i \Delta_i$ with $a_i \in [0, 1] \cap \mathbb{Q}$.

Cox ring: Let $D_1, \ldots, D_k$ be $\mathbb{Q}$-divisors. on a normal variety X. We want to define a ring

$$R(X, D^{\cdot}) = \bigoplus_{\vec{m}\in\mathbb{N}^k} H^0(X, \mathcal{O}_X(\lfloor m_i D_i \rfloor)).$$

(To round down a divisor, write it as a combination of prime divisors, and round down each coefficient.) A *log pair* is a pair (X, Δ) with $\Delta = \sum a_i \Delta_i$ with $a_i \in [0, 1] \cap \mathbb{Q}$ such that

$K_X + \Delta$ is $\mathbb{Q}$-Cartier. Call (X, Δ) log smooth if X is smooth and $\operatorname{supp}(\Delta)$ has global normal crossings. Global normal crossings mean that all intersections are smooth.

One conjecture to rule them all:

Conjecture 5.6. Let X be a projective variety. Let (X, Δ_i) be smooth pairs, $i = 1, \ldots, k$. Let $D_i = K_X + \Delta_i$. Then $R(X, D^{\cdot})$ is finitely generated.

Example 5.7. Let $X = \mathbb{P}^1$. Take $k = 1$. Let $\Delta = \frac{p}{2} + \frac{3q}{4} + \frac{5r}{6}$, where p, q, r are distinct points of X. Consider

$$\bigoplus_{m \in \mathbb{N}} H^0(\mathbb{P}^1, \mathcal{O}(\lfloor mK_{\mathbb{P}^1} + \frac{m}{2}p + \frac{3m}{4}q + \frac{5m}{6}r \rfloor)).$$

This is

$$\bigoplus_{m \in \mathbb{N}} H^0(\mathbb{P}^1, \mathcal{O}(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{3m}{4}q \rfloor + \lfloor \frac{5m}{6}r \rfloor - 2m)).$$

We have $R_0 = \mathbb{C}$, $R_m = 0$ for $1 \le m < 4$, $R_4 = \mathbb{C}E_4$, $R_5 = 0$, $R_6 = \mathbb{C}E_6$, and R_{12} is generated by E_4^3 and E_6^2. It turns out the even-degree subring $R_{(2)}$ is $\mathbb{C}[E_4, E_6]$. But $R_{11} \neq 0$!

Conjecture 5.6 implies the minimal model conjecture (existence of a geometrically meaningful compactification), termination of flips (MMP with scaling), abundance (if K_X is nef, it is semiample).

Part by part: Suppose that $R(X, K_X) \neq \mathbb{C}$. Then take $Y = \operatorname{Proj} R(X, K_X)$. We get $X \overset{\phi}{\dashrightarrow} Y$. This is birational if K_X is big (this is the definition of big).

Finite generation of the Cox ring holds if and only if finite generation holds for $k = 1$ and fix the support of Δ, i.e., $\Delta = \sum a_i \Delta_i$ with Δ_i irreducible and $a_i \in [0, 1]$, then $\{\operatorname{Proj} R(X, K_X + \Delta)\}$ is finite and the decomposition into pieces is rational polyhedral.

Example 5.8. Let $R = \mathbb{C}[x, y]$. Associate to $x^i y^j$ the point (i, j). In $\mathbb{N}^2$, take a line of slope m. Take $I := \{(i, j) : j \le mi\}$. Take $S \subseteq R$ be the span of $x^i y^j$ with $(i, j) \in I$.

For example, if $m = 3/2$, then $S = \mathbb{C}[x, xy, x^2y^3]$.

But if $m = \sqrt{2}$, the algebra S is not finitely generated.

6. MMP with scaling

Suppose D is a divisor such that

(1) $K_X + \Delta + tD$ is nef for some $t \ge 0$
(2) $K_X + \Delta + D$ is Kawamata log terminal (klt); e.g. log smooth
(3) $K_X + \Delta$ is big.

(1) Pick t minimal such that $K_X + \Delta + tD$ is nef.
(2) If $t = 0$, then STOP.
(3) Otherwise there exists a $K_X + \Delta$-negative extremal contraction $\pi \colon X \to Z$, always birational. Two cases:
   (a) exceptional locus is a divisor; replace X by Z, go back to (1)
   (b) π is small; replace X by X^+ (flips exist by finite generation).

Why do we get termination? Each time you flip, you end up in a new chamber.

Why should one believe conjecture A? Suppose H is ample on X. Why should $R(X, H)$ be finitely generated? We have

$$0 \to \mathcal{O}_X((m-1)H) \to \mathcal{O}_X(mH) \to \mathcal{O}_H(mH) \to 0,$$

and by induction one could assume that $R(H, H|_H)$ is finitely generated. If we had

$$0 \to H^0(\mathcal{O}_X((m-1)H)) \to H^0(\mathcal{O}_X(mH)) \to H^0(\mathcal{O}_H(mH)) \to 0,$$

then we would have

$$R(X, H) \to R(H, H|_H) \to 0$$

one could lift generators, etc. In general:

(1) Use method of multiplier ideal sheaves, due to Siu and Kawamata, to lift certain sections.
(2) Shokurov has shown that the image of $R(X, H) \to R(H, H|_H)$ is quite special, satisfies saturation.

There is also nice work of Eyssidieux, Guedj, and Zeriahi, proving that a canonical model has a singular Kahler-Einstein metric. It might be possible to push similar techniques to prove finite generation, without assuming the existence of a canonical model.

University of California at Santa Barbara