

# SUPERCHARACTERS AND ALGEBRA GROUPS

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- (1) Characters? Motivation: Let  $G$  be a finite group. Let  $S$  be a generating set. Then we can form the Cayley graph whose set of vertices is  $G$ , and whose edges are group elements connected by an element of  $S$ . A long enough random walk on the Cayley graph tends to the uniform distribution on  $G$ . More precisely, if  $S = S^{-1}$  and  $\text{id} \in S$ , and  $Q(g) := \begin{cases} \frac{1}{|S|}, & \text{if } g \in S \\ 0, & \text{otherwise} \end{cases}$ , then  $Q^{*h}(g) \rightarrow \frac{1}{|G|}$  as  $h \rightarrow \infty$ , where  $Q^{*h}$  is the iterated

convolution. How large must  $h$  be to make  $\|Q^{*h} - U\| := \sum \left| Q^{*h}(g) - \frac{1}{|G|} \right| < \epsilon$ ? For  $G = S_n$  and  $S = \{\text{id}, (12), (123 \dots n), (n \dots 321)\}$ , one needs  $h$  about  $n^3 \log n$ .

For general finite groups, if  $S$  is a conjugacy class  $C$ , then

$$\|Q^{*h} - U\| := \sum_{\rho} d_{\rho}^2 \left( \frac{\chi_{\rho}(C)}{d_{\rho}} \right)^{2h}.$$

- (2) Example. Let  $G$  be the set of upper triangular  $n \times n$  matrices over  $\mathbb{F}_p$  with 1s on the diagonal. Walk: pick a row at random other than the first, and add it to the row above it. It is unknown how long this takes to get random: its order of growth is between  $n^2$  and  $n^3 \log n$ , and probably the lower estimate is correct.
- (3) Problem: No nice description of the conjugacy classes and characters is known.
- (4) Work of Andre, Carter, Yan gives unions of classes and sums of irreducibles that are describable and useful.
- (5) Let  $G$  be a finite group. Let  $K$  be a partition of  $G$  in which each part is a union of conjugacy classes. Let  $\mathcal{X}$  be a partition of the set  $\text{Irr}(G)$  of irreducible representations of  $G$ . For  $X \in \mathcal{X}$ , let  $\psi_X$  be a character whose constituents are contained in  $X$ .

Assume:

- (a)  $\text{id} \in K$
- (b)  $|K| = |\mathcal{X}|$
- (c) The  $\psi_X$  are constant on elements of  $K$ .

Definition: This is a supercharacter theory.

Examples:

- (a)  $K = \{\{\text{id}\}, G - \{\text{id}\}\}$ ,  $\mathcal{X} = \{\{X_{\text{triv}}\}, \text{Irr}(G) - \{X_{\text{triv}}\}\}$
- (b)  $K$  the set of all conjugacy classes,  $X = \text{Irr}(G)$
- (c)  $G$  any group,  $A \leq \text{Aut}(G)$ ,  $A$  acts on  $\text{Irr}(G)$  and on conjugacy classes; let  $X$  and  $K$  be the set of orbits, respectively. E.g., consider  $G = C_2^n$  and  $A = S_n$ .

**Theorem 0.1.** *Given a supercharacter theory,*

- (a)  $\chi_X = c(X) \sum_{\chi \in X} \chi(\text{id}) \chi(\cdot)$  for some  $c(X) \in \mathbb{Z}$ .
- (b)  $\{\psi_X\}$  is a basis for the set of functions constant on the elements of  $K$ .

(c) *Each of  $K$  and  $X$  determines the other.*

One  $S_3$ , there are only the two trivial supercharacter theories. For  $S_6$ , there is one nontrivial one, coming from the outer automorphism. For  $S_n$  in general, we don't know.

- (6) Algebra groups. Let  $J$  be a finite-dimensional nilpotent associative algebra (without 1) over  $\mathbb{F}_q$ . Let  $G$  be the set of elements  $\{1+x\}$  with  $(1+x)(1+y) := 1+x+y+xy$ . Such a  $G$  is called an *algebra group*.

Examples:

- (a) Let  $J$  be set of elements  $\sum_{i=1}^n a_i x^i$  in  $\mathbb{F}_q[x]/(x^{n+1})$ . Then  $G$  is the group of elements  $1 + \sum_{i=1}^n a_i x^i$ .
- (b) Let  $J$  be the strictly upper triangular matrices over  $\mathbb{F}_q$ .
- (c) Given a partial ordering on  $\{1, \dots, n\}$ , take the subgroup  $G$  in which the  $(i, j)$ -entry is allowed to be nonzero if and only if  $i \leq j$ .
- (7) Superclasses for algebra groups. The group  $G \times G$  acts on  $J$  (multiplication on left and right). Then  $\{1 + GxG : x \in J\}$  is a partition of  $G$ . In the examples before, we have
- (a)  $\text{SC}(i) = \{1 + c_i x^i + (\text{higher order terms})\}$  has size  $q^{n-i}$
- (b) Orbits are indexed by set partitions. e.g., for  $n = 3$ :

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & x \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & x & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

- (8) Supercharacters.

$$T^* := \{\lambda: J \rightarrow \mathbb{F}_q\}.$$

On this  $G$  acts. Choose a homomorphism  $\mathbb{F}_q \rightarrow \mathbb{C}^*$ , say  $\alpha \mapsto \tilde{\alpha}$ . For  $y = 1+x$ , define  $\tilde{\lambda}(y) := \lambda(\tilde{x})$ . The linear span of  $u$  such that  $\tilde{u} \in (-\lambda)G$  is a  $G$ -module; let  $\chi_\lambda$  be its character.

**Proposition 0.2.** *This  $\chi_\lambda$  depends only on the  $G(-\lambda)G$  orbit and these are supercharacters.*

- (9) Nice theorems:

- (a) Restriction of supercharacters to all subgroups.
- (b) Tensor product of two is a supercharacter; positive integer combination of supercharacters is a supercharacter.
- (c)  $\psi_H^{(G)}(1+x) = \frac{1}{|G||H|} \sum_{g_1, g_2 \in U} \psi^U(1 + g_1 x g_2)$ .
- (d)  $[\phi^{(G)}, \theta] = [\phi, \theta_H]$

- (10) Nice formulas. E.g., for upper triangular matrices with 1s on the diagonal., character  $\leftrightarrow B, \phi$ , conjugacy class  $\leftrightarrow B', \phi'$ ,

$$\begin{cases} q^{p(B, B')} \prod_{(i, j) \in B \cap B'} \phi(i, j) \phi'(i, j), & \text{if } B \subseteq R(B') \\ 0, & \end{cases}$$

where  $R(B')$  is the complement of boxes above  $B$  with  $B' \subset R(B')$ , and  $P(B, B')$  is the number of positions directly below positions in  $B$  also in  $R(B')$ .

## REFERENCES

- [1] 1. Arias-Castro, E. Diaconis, P. and Stanley, R. "A Super-Class Walk on Upper-Triangular Matrices". Jour. Alg. 278 (2004), 739-765.
- [2] 2. Diaconis, P. and Isaacs, I.M. (2006), "Super-Characters and Superclasses for Algebra Groups". To appear Trans. Amer. Math. Soc.

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