BOUNDEDNESS OF FAMILIES OF CANONICALLY POLARIZED VARIETIES: A HIGHER-DIMENSIONAL ANALOGUE OF SHAFAREVICH'S CONJECTURE

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1. A Shafarevich conjecture

We work over $\mathbb{C}$. Let B be a smooth curve of genus g. Let $\Delta \subset B$ be a finite set of closed points. (This is roughly analogous to the situation in which B is the ring of integers in a number field, and Δ is a finite set of primes.)

Question: What kinds of smooth families of curves are there over $B - \Delta =: B^0$.

Definition 1.1. A *family* is a proper flat $X^0 \to B^0$ with connected fibers.

Example: Take any fixed curve over $\mathbb{C}$, and take its product with B^0.

Definition 1.2. A family is isotrivial if for all $b_1, b_2 \in B^0(\mathbb{C})$, $X^0_{b_1} \simeq X^0_{b_2}$.

Conjecture 1.3 (Shafarevich, 1962)**.** Given B and Δ and $h \geq 2$, then

(1) The set of non-isotrivial families over $B - \Delta$ with fibers of genus h is finite.
(2) If $2g - 2 + \#\Delta \leq 0$, there are no families.

Parshin (1968):

(1) Proved the conjecture for $\Delta \neq \emptyset$.
(2) Proved that the conjecture implies the Mordell conjecture: If C is non-isotrivial of genus ≥ 2, then it has finitely many points.

Arakelov (1971): Proved the full conjecture.

Caporaso (2002, 2003): Proved that there exists a uniform bound on the number of families in terms of g, $\#\Delta$, h. She also consider families over higher-dimensional bases.

Heier (2004): Effective bound.

Strategy of proof: Two subproblems:

(1) Boundedness: there are finitely many deformation types.
(2) Rigidity: Families admit no non-trivial deformations.

Definition 1.4. A *deformation* of families over T is a family $X \to B^0 \times T$. A *deformation type* is an equivalence class for the relation defined by deformations over connected T.

Question: What about higher-dimensional fibers? And higher-dimensional bases?

Let B^0 be an arbitrary smooth variety.

Definition 1.5. An *admissible family* is a proper smooth morphism $f\colon X^0 \to B^0$ such that for all $b \in B^0$, $\omega_{X^0_b}$ is ample on X^0_b.

Date: April 4, 2006.

Hilbert polynomial: $h(n) := \chi(\omega_{X_b^0}^{\otimes n})$ has $\deg h = \dim X_b^0$.

Theorem 1.6 (Kovács, Lieblich). *Fix an integer-valued polynomial h of degree n. Assuming* MMP$(n+1)$*, the minimal model program for $(n+1)$-dimensional varieties,*

(1) *The set of deformation types of admissible families with Hilbert polynomial h is finite.*
(2) *The cardinality is bounded in any quasi-compact family of bases B^0.*

If B^0 is proper, then we do not need MMP.

Remark 1.7. MMP(3) is proven.

Theorem 1.8 (Bedulev-Viehweg). *Given a smooth projective curve B, a finite subset $\Delta \subset B$, and a polynomial h, if $f\colon X \to B$ is a proper flat family such that X is smooth and $f|_{B-\Delta}$ is admissible with Hilbert polynomial h, then for $m \gg 0$,*

$$\deg\left(f_*\omega_{X/B}^{\otimes m}\right) \leq \textit{polynomial function of } g(B),\ \#\Delta,\ m,\ \textit{coefficients of } h.$$

2. Moduli interpretation

Theorem 2.1. *Assuming* MMP$(n+1)$*, fix a polynomial h, there exists a diagram*

$$\begin{array}{ccccc} & & \overline{\mathcal{M}}_h & \longleftarrow & \mathcal{M}_h \\ & & \downarrow & & \downarrow \\ \mathbb{P}^N & \longleftarrow & \overline{M}_h & \longleftarrow & M_h \end{array}$$

where horizontal maps are open immersions, with coarse moduli spaces on the bottom, in which

(1) *$\overline{\mathcal{M}}_h$ is a proper DM stack parameterizing stable varieties with Hilbert polynomial h*
(2) *$\mathcal{M}_h$ is an open substack parameterizing smooth canonically polarized varieties with Hilbert polynomial h*
(3) *If $\mathcal{X} \xrightarrow{\phi} \overline{\mathcal{M}}_h$ is the universal variety, then*

$$\left(\det \phi_*\omega_{\mathcal{X}/\overline{\mathcal{M}}_h}\right)^p$$

is the pullback of a very ample invertible sheaf on $\overline{M}_h$.

Given $X^0 \to B^0$, we get $B^0 \to \mathcal{M}_h$ in $\mathbf{Hom}(B^0, \mathcal{M}_h)$, from which we get $B^0 \to M_h$ in $\mathbf{Hom}(B^0, M_h)$.

3. Compact bases (no MMP)

Given $X \to B$, we get $\phi_X\colon B \to M_h$, and M_h carries the $\mathcal{O}(1)$ given by the theorem.

Bedulev-Viehweg implies that $\phi_X^*\mathcal{O}(1)$ lies in a quasi-compact subscheme of Pic_B, so $\phi_X \in T \subset \mathbf{Hom}(B, M_h)$ where T is finite type. The map $\mu\colon \mathbf{Hom}(B, \mathcal{M}_h) \to \mathbf{Hom}(B, M_h)$ factors through T.

Theorem 3.1 (Olsson). *The map μ is of finite type.*

Thus $\mathbf{Hom}(B, \mu_h)$ is of finite type.

Lemma 3.2. *The set of deformation types equals the set of connected components of* $\mathbf{Hom}(B, \mathcal{M}_h)$.

4. Non-compact bases

$X^0 \to B^0$, $B^0 \to \mathcal{M}_h$. The map $B^0 \stackrel{\phi_{X^0}}{\to} M_h \subset \overline{M}_h$ extends to $B \to \overline{M}_h$ for some B (depending on ϕ_{X^0}) where $B - B^0$ is snc.

Given ϕ_{X^0}, there exists a stack $\tilde{B}$ with coarse moduli space B having an open substack isomorphism to B^0 such that every family over B^0 extends to a stable family over $\tilde{B}$.

$\mathbf{Hom}(\tilde{B}, \overline{\mathcal{M}}_h) \to \mathbf{Hom}(\tilde{B}, \overline{M}_h)$ is of finite type, and the **Hom** on the right has a finite type piece containing coarse maps.

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