AN INTRODUCTION TO STACKS

ANDREW KRESCH

1. Motivation

Two motivating problems:

(1) Connectedness of the moduli space M_g of curves of genus $g \geq 2$ over an arbitrary base field. This was proved by Deligne and Mumford in 1969. They introduced stacks to do this.

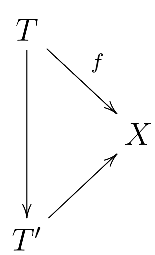
Outline of their stack-based proof: They wanted to use the Enriques-Zariski connectedness theorem: in a smooth proper family of algebraic varieties, the number of connected components of geometric fibers remains constant. But for $M_g \to \operatorname{Spec} \mathbb{Z}$ it doesn't quite work, since it is not proper. However, one can work with the compactification by stable curves $\overline{M}_g$, which is projective. It is also not smooth, but this moduli problem is smooth at the level of deformation theory. Introduce an algebraic stack $\overline{\mathcal{M}}_g \to \operatorname{Spec} \mathbb{Z}$ that is both smooth and proper. Then prove a stack version of the connectedness theorem. It was known classically that M_g over $\mathbb{C}$ was connected. Hence M_g is connected over any base field.

(2) $\operatorname{Br} X = \operatorname{Br}' X$, where $\operatorname{Br} X$ parameterizes sheaves of Azumaya algebras up to equivalence (these generalize central simple algebras over a field), and $\operatorname{Br}' X := H^2(X, \mathbb{G}_m)_{\text{tors}}$. There is a natural injection $\operatorname{Br} X \to \operatorname{Br}' X$, and the question is whether it is surjective. In the 1990s, equality was proved by Gabber for any scheme with an ample invertible sheaf. There is a new proof by de Jong using stacks.

Idea: We have $H^2(X, \mu_n) \to H^2(X, \mathbb{G}_m)$ mapping onto the n-torsion subgroup. Given $\beta \in H^2(X, \mathbb{G}_m)$, we can lift β to $\alpha \in H^2(X, \mu_n)$. Then α corresponds to a *gerbe* $\mathcal{X}$: a kind of algebraic stack. Then β comes from a sheaf of Azumaya algebras if and only if a certain kind of locally free sheaf exists on $\mathcal{X}$.

2. Definition of algebraic stack

Consider a base category: for us, the category of schemes. For any scheme X, we have its *functor of points* $h_X = \operatorname{Hom}(-, X)$. We get a category $\underline{X}$ in which an object is a scheme T equipped with a map to X, and a morphism from $T \to X$ to $T' \to X$ is a morphism $T \to T'$ mapping



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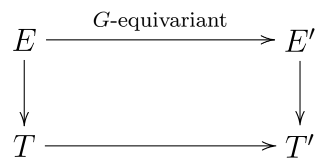
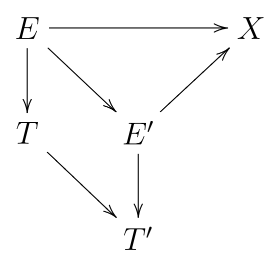
There is a functor from $\underline{X}$ to the category of schemes, taking $T \to X$ to T, and a morphism from $T \to X$ to $T' \to X$ to $T \to T'$. This is an example of a category fibered in groupoids

Another example: Consider an algebraic group G over a field k, or a finite group G, or a group scheme $G \to S$. Let BG be the category of G-torsors: an object consists of $E \to T$ equipped with a right G-action on E respecting the map to T, such that étale locally it is isomorphic to $T \times G \to T$ with the obvious action of G, and a morphism from $E \to T$ to $E' \to T'$ is a commutative diagram



There is a functor from BG to the category of schemes sending $E \to T$ to T, and a morphism as above to $T \to T'$.

A third example, combining the previous two: Let X be a scheme with a right G-action. Then $[X/G]$ is the category whose objects are torsors $E \to T$ with a G-equivariant map to X, and whose morphisms are diagrams

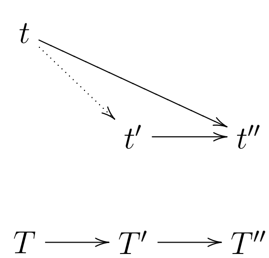


Again there is a functor from $[X/G]$ to the category of schemes.

Definition 2.1. A category fibered in groupoids is a category plus a functor to a base category such that

(i) "Pullbacks exist": Given a morphism $T \to T'$ in the base category and an object t' over T', there exists t over T with a morphism $t \to t'$ over $T \to T'$,

(ii) "and are unique up to canonical isomorphism": Given



there exists a unique dotted arrow lying over the given $T \to T'$ and making the triangle commute.

E.g., BG is a category fibered in groupoids:

$$\begin{array}{ccc} T \times_{T'} E & \longrightarrow & E' \\ \downarrow & & \downarrow \\ T & \longrightarrow & T' \end{array}$$

Existence and universal property of fiber product.

In BG, there can be many morphisms over a given morphism in the category of schemes. For instance, for a torsor $E \to T$, above the identity $1_T \colon T \to T$ one has translation-by-g from $E \to E$ for any $g \in G$. This is a feature not present in the example $\underline{X}$, and is what makes stacks "stacky".

If G acts freely on X, then there is at most one morphism over a given morphism in $[X/G]$.

E.g., let $G = \mathbb{Z}/2$ act on $\mathbb{A}^2_{\mathbb{C}}$ by $(x, y) \mapsto -(x, y)$. Then $[\mathbb{A}^2/G]$ is a cone with a stacky $\mathbb{Z}/2$-point at the origin. The quotient variety $\operatorname{Spec} \mathbb{C}[t, u, v]/(t^2 - uv)$ is singular (it's a cone), even though the quotient stack will turn out to be smooth. The map $\mathbb{A}^2 \to \operatorname{Spec} \mathbb{C}[t, u, v]/(t^2 - uv)$ is given by $(x, y) \mapsto (xy, x^2, y^2)$.

Let $\mathcal{X}$ be a category fibered in groupoids. Let $\mathcal{X}_T$ be the category whose objects are objects over T, and whose morphisms are morphisms over 1_T. The axioms for a category fibered in groupoids imply that $\mathcal{X}_T$ is a groupoid, i.e., a category in which all morphisms are isomorphisms. It is "stacky" for some T if some objects of $\mathcal{X}_T$ have more than one isomorphism.

Equip the base category (schemes) with a topology, the *étale topology*: a collection of morphisms $T_i \xrightarrow{\phi_i} T$ is a covering if each ϕ_i is étale, and T is covered by the $\phi_i(T_i)$.

Definition 2.2. A *stack* is a category fibered in groupoids satisfying two gluing axioms:

(1) Gluing for morphisms: Given a covering family $(T_i \to T)$ and u, v over T, and pullbacks u_i, v_i of u, v to objects over T_i and $u_i \xrightarrow{f_i} v_i$ over 1_{T_i}, and $T_{ij} := T_i \times_T T_j$ such that

$$\begin{array}{ccc} u_i|_{T_{ij}} & \xrightarrow{f_i|_{T_{ij}}} & v_i|_{T_{ij}} \\ \| & & \| \\ u_j|_{T_{ij}} & \xrightarrow{f_j|_{T_{ij}}} & v_j|_{T_{ij}} \end{array}$$

(2) Gluing for objects: Given T, $(T_i \to T)$, u_i over T_i, isomorphisms $u_i|_{T_{ij}} \xrightarrow{\phi_{ij}} u_j|_{T_{ij}}$ satisfying a cocycle condition, there exists u over T and isomorphisms $u|_{T_i} \to u_i$ over 1_{T_i}.

Example 2.3. BG, where G is an affine group scheme (or linear algebraic groups over a field, or a finite group). Gluing for morphisms is clear. Gluing for objects relies on a theorem. Given torsors $E_i \to T_i$, if G is affine we can write $E_i = \mathbf{Spec}\, \mathcal{E}_i$; now use descent for quasi-coherent sheaves of modules to get $\mathcal{E}$ over T, and let $E = \mathbf{Spec}\, \mathcal{E}$.

Warning: If E is an elliptic curve, then BE is not a stack, unless one generalizes the definition and works with algebraic spaces in place of schemes.

Example 2.4. For any scheme X and any affine group scheme G acting on X, the quotient $[X/G]$ is a stack.

3. Algebraic stacks

Next we will define

- algebraic stacks,
- algebraic spaces (an algebraic space is a non-"stacky" algebraic stack).

Example 3.1. Let $g \geq 2$. Let $\overline{\mathcal{M}}_g$ be the category whose objects are families of stable genus-g curves $C \to T$ and whose morphisms are cartesian diagrams

$$\begin{array}{ccc} C & \longrightarrow & C' \\ \downarrow & & \downarrow \\ T & \longrightarrow & T' \end{array}$$

We claim that this is a stack. Gluing morphisms is easy. To glue objects, one needs to show that $\omega_{C/T}$ is relatively ample, and use a construction involving **Proj** just as **Spec** was used above.

Deligne and Mumford gave a definition of algebraic stack. These satisfy the following axioms:

(1) Representability of the diagonal.
(2) Existence of an étale cover by a scheme.

These are now called Deligne-Mumford stacks, since Artin in 1974 gave a more general definition, and these are now called Artin stacks.

Given u, v over T, let $\mathbf{Isom}_T(u, v)$ be the presheaf whose sections over $T' \to T$ the set of morphisms $u' \to v'$ in $\mathcal{X}_{T'}$ between pullbacks u', v' over T' of u, v (this is independent of the choice of u', v'). The axiom "Representability of the diagonal" says that $\mathbf{Isom}_T(u, v)$ is representable by a scheme that is separated and of finite type over T.

Each scheme X can be viewed as a stack $\underline{X}$.

A morphism of stacks is a functor commuting with the functors to the base category of schemes. One can define a fiber product of categories fibered in groupoids over another category fibered in groupoids. Then one can interpret $\mathbf{Isom}_T(u, v)$ as the fiber product

$$\begin{array}{ccc} \mathbf{Isom}_T(u, v) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ T & \longrightarrow & \mathcal{X} \times \mathcal{X} \end{array}$$

Given schemes T, U with morphisms to $\mathcal{X}$, we have a cartesian diagram

$$\begin{array}{ccc} \mathbf{Isom}_{T \times U}(p_1^* t, p_2^* u) & \longrightarrow & U \\ \downarrow & & \downarrow \\ T & \longrightarrow & \mathcal{X}. \end{array}$$

Thus the axiom "Representability of the diagonal" implies that arbitrary fiber products of schemes over a stack $\mathcal{X}$ exist as schemes.

Say that $U \to \mathcal{X}$ is (étale, surjective, flat, smooth) if $T \times_{\mathcal{X}} U \to T$ is, for all $T \to \mathcal{X}$.

Definition 3.2. A *Deligne-Mumford (DM) stack* is a stack $\mathcal{X}$ with representable separated finite-type diagonal such that there exists a scheme U and an étale surjective morphism $U \to \mathcal{X}$.

A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of DM stacks is (étale, surjective, flat, smooth) if and only if the composition $U \to \mathcal{X} \xrightarrow{f} \mathcal{Y}$ is, for one (or any) U as in the definition of DM stack for $\mathcal{X}$. Warning: one cannot add "proper" to the list of adjectives above.

Examples 3.3.

- $\underline{X}$ for any quasi-separated scheme (e.g., anything of finite type over a noetherian base).
- BG, where G is an affine étale group scheme.
- $[X/G]$ where G is an affine smooth group scheme acting with finite reduced stabilizer group schemes (at geometric points).

Let us check representability of the diagonal for BG, where G is affine étale group scheme. Given torsors E_1, E_2 over T, choose an étale cover $T' \to T$ that trivializes E_1, E_2; then descend $\mathbf{Isom}_{T'}(T' \times G, T' \times G) = T' \times G$ to T. The other axiom, about existence of an étale cover by a scheme, is easy: take the base scheme $S \to BG$, where S has the trivial G-action.

For more general X/G, one needs a slice argument: i.e., a slice of $X \to X/G$ serves as the étale cover.

For an Artin stack, the diagonal should be represented by an algebraic space, and there should exist a smooth cover by a scheme. An example is $[X/G]$, where G is a smooth separated group scheme of finite presentation.

Suppose G is not smooth, but only flat. Then BG and $[X/G]$ will still be Artin stacks, as long as one works with fppf-torsors instead of étale-torsors in their definition. In this case, existence of smooth cover by a scheme is difficult theorem in Artin's paper.

Example 3.4. Consider $\mu_n \to \operatorname{Spec} \mathbb{Z}$. Then $\operatorname{Spec} \mathbb{Z} \to B\mu_n$ is not a smooth cover, but $\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m$ is a μ_n-torsor, giving rise to a morphism $\mathbb{G}_m \to B\mu_n$ that is smooth and surjective.

Use $\overline{\mathcal{M}}_g = [U_g/\operatorname{PGL}_n]$ to show that $\overline{\mathcal{M}}_g$ is a DM stack. More recently, it has been proved that one can write $\overline{\mathcal{M}}_g = [V/G]$ with G finite.

4. Gerbes

Fix a group G (finite, or μ_n, or $\mathbb{G}_m$). A *gerbe* is an algebraic stack $\mathcal{X}$ over a scheme X that is étale locally a product with BG. In fact, X is determined by $\mathcal{X}$.

Define $I_{\mathcal{X}}$ as the fiber product

$$\begin{array}{ccc} I_{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X} \end{array}$$

Then given $T \to \mathcal{X}$, one gets

$$\begin{array}{ccccc} \mathbf{Aut}_T(u) & \longrightarrow & I_{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X} \end{array}$$

5. Structure results

Over a base field k, lots of finite-type smooth DM stacks (e.g., those with trivial generic stabilizer, or under weaker hypotheses when $\operatorname{char} k = 0$) are of the form $[X/G]$ where X is a scheme or algebraic space and G is an algebraic group.

University of Warwick and MSRI