STRANGE SURFACES IN POSITIVE CHARACTERISTIC

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Suppose we are in characteristic 0. Classify using some invariants such as K_X^2, χ. Today: K_X^2/χ. Are there restrictions on the possible invariants?

Van de Van: $K_X^2/\chi \leq 10.666\cdots$. Conjectured ≤ 9. There are examples with $= 9$:

(1) $\mathbb{P}^2$.
(2) Hirzebruch (1958)
(3) Borel (1963)

Bogomolov (1976) proved $K^2/\chi \leq 9.6$. He used a new notion of instability of rank 2 vector bundles.

Miyaoka (1977) and Yau (1977), using ideas of Bogomolov, proved ≤ 9. See Miyaoka's proof in the book by Barth, Peters, Van de Ven.

What happens in positive characteristic? The proofs of Bogomolov and Miyaoka use lemmas that are known to fail in characteristic p. Bill Lang (1983) gives "generalized Raynaud surfaces" such that $K^2/\chi > 12$. (The bound tends to 12 as $p \to \infty$.)

Hirzebruch (1984) gets more examples with $= 9$ in characteristic 0. Two ingredients:

(1) configurations of lines in the plane: 3 extreme configurations
(2) abelian branched covers of surfaces

Example of a pathological configuration: Consider the $\mathbb{F}_2$-valued points in $\mathbb{P}_k^2$. Look at the 7 lines through these points.

Use abelian branched covers (Catanese, Manetti with $(\mathbb{Z}/2)^n$; Pardini with all abelian groups).

Idea: Let G be a finite abelian group. A finite map $\pi\colon \tilde{X} \to X$ is an abelian G-cover if there exists a G-action on $\tilde{X}$ with $X \simeq \tilde{X}/G$; i.e., the function field extension is Galois with Galois group G.

Building data: Some invertible sheaves $\mathcal{L}_\chi$ on X indexed by characters $\chi \in \hat{G}$ and some divisors D_σ on X labelled by $\sigma \in G$, satisfying a covering condition. Then we get an algebraic structure on $\mathbf{Spec} \bigoplus_\chi \mathcal{L}_\chi$.

Our case (notation from Vakil): Let $G = (\mathbb{Z}/q)^n$ where q is a prime. Work over k of positive characteristic $p \neq q$. Let

$$\langle\ \ \rangle\colon G \times \hat{G} \to k^\times$$

be the standard pairing. Choose $\mathbb{Z} \to k^\times$ mapping onto the q-th roots of 1, and lift to a (no longer bilinear) pairing

$$\langle\ \ \rangle\colon G \times \hat{G} \to \mathbb{Z}.$$

Let $D\colon G \to \operatorname{Div}(X)$ and $L\colon \hat{G} \to \operatorname{Pic}(X)$ be maps. Let X be the blowup of $\mathbb{P}_k^2$ at all $\mathbb{F}_p$-points of $\mathbb{P}^2$.

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The covering condition is

$$qL_\chi \sim \sum_\sigma \langle \sigma, \chi \rangle [D_\sigma],$$

where we are now writing the group law in $\mathrm{Pic}(X)$ additively.

Theorem 0.1 (Catanese, Pardini)**.** *Suppose D_σ, L_χ are building data satisfying the covering condition such that all the D_σ are nonsingular curves, and no three meet at a point, and if $D_\sigma, D_{\sigma'}$ meet, they meet transversely and σ and σ' are $\mathbb{F}_q$-independent in G. Then*

1. *There is a branched G-cover $\pi\colon \tilde{X} \to X$ that is nonsingular and branched over D_σ.*
2. $q^n K_{\tilde{X}} \sim \pi^*(q^n K_X + q^{n-1}(q-1)\sum_\sigma D_\sigma)$ *(use this to check that $\tilde{X}$ is of general type)*
3. $K_{\tilde{X}}^2 = q^n \left(K_X + \frac{q-1}{q}\sum D_\sigma\right)^2$
4. $\chi(\tilde{X}) = q^n \chi(X) + \frac{1}{2}\sum_\chi L_\chi(L_\chi + K_\chi)$

Our application. Let $G = (\mathbb{Z}/q)^n$ with $q, n \geq 3$. Let $C \subseteq \mathbb{P}^2$ be the union of the lines through the pairs of $\mathbb{F}_p$-valued points. Let $\tilde{C} \subseteq X$ be the strict transform of C. Let $\sigma_1 = (1, 0, 0, \dots, 0)$. Choose $D_0 = 0$ and $D_{\sigma_1} = \tilde{C}$. Hope that for all $\sigma \in G - \{0, \sigma_1\}$, we can choose $D_\sigma \sim a_\sigma[H]$ for some good $a_\sigma \in \mathbb{Z}$.

The only nontrivial part is arrange that if $a_\sigma \neq 0$, then $a_{k\sigma} = 0$ for all $k \neq 1$.

Fact: there is a choice of a_σ such that all the conditions hold.

Good fact: Suppose that q, n are fixed. Then for any choice of a_σ that works, we have estimates:

$$K^2 = q^{n-2}(q^2-1)p^3 + O(p^2)$$

$$\chi(\tilde{X}) = \frac{1}{12} q^{n-2}(q^2-1)p^3 + O(p^2).$$

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