MODULI OF ABELIAN VARIETIES AND THEIR COMPACTIFICATIONS

MARTIN OLSSON

$$\begin{tikzcd}
\mathcal{M}_{1,1} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{1,1} \arrow[d] \\
\mathbb{A}^1_j \arrow[r] & \mathbb{P}^1_j
\end{tikzcd}$$

Here $\overline{\mathcal{M}}_{1,1}$ is the moduli space of flat families $C \to T$ with a section such that the geometric fibers are either an elliptic curve or a nodal cubic.

There are two possible generalizations: First, the moduli space $\mathcal{M}_{g,n}$ has a compactification $\overline{\mathcal{M}}_{g,n}$, the Deligne-Mumford-Knudsen compactification. Second, $\mathcal{A}_g$ is the moduli space of abelian varieties of dimension g with principal polarization. (An abelian variety is a complete variety with a group structure.)

Question 0.1. How to compactify $\mathcal{A}_g$?

What properties should the compactification $\mathcal{A}_g \subset \overline{\mathcal{A}}_g$ have?

(a) $\overline{\mathcal{A}}_g$ should be proper over $\mathbb{Z}$, and $\mathcal{A}_g \subset \overline{\mathcal{A}}_g$ should be dense open.
(b) $\overline{\mathcal{A}}_g$ should represent a "simple" functor.
(c) $\overline{\mathcal{A}}_g$ should be (nearly) smooth, $\overline{\mathcal{A}}_g - \mathcal{A}_g$ divisor with normal crossings (i.e., it is log smooth). One could then blow-up to make it smooth, but one would lose the simple modular meaning.
(d) There should be an analytic theory at $\partial\overline{\mathcal{A}}_g$, generalizing the Tate curve.
(e) There is a combinatorial description of $\partial\overline{\mathcal{A}}_g$.
(f) There should be variants with level structure and higher degree polarizations.

History:

- Satake compactification (1956): this was singular at the boundary.
- Toroidal compactifications
  - Mumford et al (1975).
  - Faltings and Chai (1990).
  - Kajiwara, Kato, Nakayama (2005).
- Alexeev (2002), Nakamura: constructed a functor that is complete but contains many components. This suggests that there should exist a canonical compactification.

Date: February 14, 2006.

1. RETHINKING $\mathcal{A}_g$

Let B be a base scheme. Let A/B be an abelian scheme. Let $P \to B$ be a principal homogeneous space under A. Now $\mathbf{Pic}(P)$ classifies invertible sheaves on P (relative to B). It has a subspace $\mathbf{Pic}^0(P)$, the connected component of the class $[\mathcal{O}_P]$ of the structure sheaf.

Remark 1.1. Suppose $P = A$. Then $\mathbf{Pic}^0(A) = A^t = \mathbf{Ext}^1(A, \mathbb{G}_m)$, the dual abelian variety. This is the functor

$$T \mapsto \{(L, \iota \colon e^*L \xrightarrow{\sim} \mathcal{O}_T)\}$$

where L is a line bundle on A_T. Given L on P, we have

$$\begin{aligned} \lambda_L \colon A &\to \mathbf{Pic}^0(P) \\ a &\mapsto [t_a^*L \otimes L^{-1}] \end{aligned}$$

Consider the set of (A, P, L, θ) such that A is an abelian variety of dimension g over B, $f \colon P \to B$ is an A-torsor, L is a line bundle on P such that $\lambda_L \colon A \to \mathbf{Pic}^0(P) = A^t$ is an isomorphism, and $\theta \in f^*L$ is nonvanishing in every fiber.

This maps to $\mathcal{A}_g(B)$.

Proposition 1.2. *This map is an equivalence.*

Upshot: One can think of $\mathcal{A}_g$ as classifying (A, P, L, θ)'s.

Idea: let A degenerate to a semiabelian variety, and let P degenerate too. We will need to add some log structure too.

2. GEOMETRIC POINTS OF $\overline{\mathcal{A}}_g$

Suppose $k = \overline{k}$. A *semiabelian variety* G/k is an extension

$$0 \to T \to G \to Ato0$$

where $T \simeq \mathbb{G}_m^r$. Let X be the character group $\mathrm{Hom}(T, \mathbb{G}_m)$. Giving an extension of A by $\mathbb{G}_m^n$ is the same as giving a homomorphism

$$c \colon X \to \mathrm{Ext}^1(A, \mathbb{G}_m) = A^t.$$

For any $x \in X$, we have a rigidified line bundle L_x on A (rigidified = equipped with a value at 0), and a canonical $L_x \otimes L_y \xrightarrow{\sim} L_{x,y}$. Then $\mathbf{Spec}_A \bigoplus_{x \in X} L_x = G$.

Let $a \colon X \to \mathbb{R}$ be a possibly non-homogeneous quadratic function with positive-definite homogeneous part. Let

$$\Gamma_a = \{(x, a(x)) : x \in X\} \subseteq X_{\mathbb{R}} \times \mathbb{R}.$$

Let S be the set of domains of linearity in $X_{\mathbb{R}}$ of the convex hull of Γ_a. So S is a collection of polytopes in $X_{\mathbb{R}}$. Note that S is X-invariant. Assume that the vertices of S are integral.

Let P be the set of integral points of $\mathrm{Cone}(1, X_{\mathbb{R}}) \subseteq \mathbb{R} \times X_{\mathbb{R}}$. For $\omega \in S$, let P_ω be the set of integral points of $\mathrm{Cone}(1, \omega)$.

Example: $A = \mathrm{pt}$, $T = \mathbb{G}_m$. We have the monoid algebra $k[(x_n)_{n \in P}]$ on P. Take the quotient R by all relations $x_n x_m = 0$ such that n, m do not lie in some P_ω. Now $\mathrm{Proj}\, R$ is an infinite chain of $\mathbb{P}^1$. It has an action of $\mathbb{Z}$, and the quotient is the nodal cubic.

In general, suppose we have

$$0 \to T \to G \to A \to 0$$

over $k = \overline{k}$. Fix a line bundle $\mathcal{M}$ on A defining a principal polarization. Let $\mathcal{R} = \bigoplus_{(d,x) \in P} \mathcal{M}^d \otimes L_x$. Define multiplication by

$$(\mathcal{M}^d \otimes L_x) \otimes (\mathcal{M}^{d'} \otimes L_y) \to \mathcal{M}^{d+d'} \otimes L_{x+y}.$$

to be 0 if (d, x) and (d', y) do not lie in some P_ω, and to be the canonical map otherwise. Let $\tilde{P} = \mathbf{Proj}_A \mathcal{R}$. To get the X-action, use symmetric biextensions. The quotient $P = \tilde{P}/X$ exists as a scheme, and comes with a line bundle L.

The (G, P, L, θ) almost correspond to boundary points. The functor $\overline{\mathcal{A}}_g$ sends B to almost the category of (G, P, L, θ) over B where G is semiabelian, $f\colon P \to B$ is proper with a G-action, L is a line bundle on P, and $\theta \in f^*L$ is nonvanishing in every fiber, such that every geometric fiber is isomorphic to

(1) A torsor under an abelian variety with L such that $\lambda_L\colon A \to A^t$ is an isomorphism, or
(2) previous construction.

Actually, the first case is a special case (with $T = \{1\}$) of the second case.

Actually, the functor sends B to the category of $(G, P, L, \theta, M_B, M_P)$ where G is semiabelian, $f\colon (P, M_P) \to (B, M_B)$ is log smooth and proper with a G-action, L is a line bundle on P, and $\theta \in f^*L$ is nonvanishing in every fiber, such that every geometric fiber is isomorphic to etc. Here M_B and M_P are log structures.

If we look at the coarse moduli space $\overline{A}_g$ (called the second Voronoi compactification by Faltings-Chai), we have

$$\begin{tikzcd}
\overline{\mathcal{A}}_g \arrow[dr] & & \overline{\mathcal{A}}_g^{\mathrm{FC}} \arrow[dl] \\
& \overline{A}_g &
\end{tikzcd}$$

University of Texas, Austin